Binomial Theorem

"Obvious" is the most dangerous word in mathematics......... Bell, Eric Temple

Binomial expression :

Any algebraic expression which contains two dissimilar terms is called binomial expression.

For example : $x + y$, $x^2y + \frac{1}{xy^2}$, $3 - x$, $\sqrt{x^2+1} + \frac{1}{(x^3+1)^{1/3}}$ etc.

Terminology used in binomial theorem :

Factorial notation : $\underline{|n}$ or n! is pronounced as factorial n and is defined as

$$n! = \begin{cases} n(n-1)(n-2)........3 \;.\; 2 \;.\; 1 & ; \text{ if } n \in N \\ 1 & ; \text{ if } n = 0 \end{cases}$$

Note : $n! = n \,.\, (n-1)!$; $n \in N$

Mathematical meaning of ^nC_r : The term ^nC_r denotes number of combinations of r things choosen from n distinct things mathematically, $^nC_r = \frac{n\;!}{(n-r)!\;\; r!}$, $n, r \in W$, $0 \le r \le n$

Note : Other symbols of of ^nC_r are $\binom{n}{r}$ and $C(n, r)$.

Properties related to ^nC_r :

(i) $^nC_r = {}^nC_{n-r}$

Note : If $^nC_x = {}^nC_y \Rightarrow$ Either $x = y$ or $x + y = n$

(ii) $^nC_r + {}^nC_{r-1} = {}^{n+1}C_r$

(iii) $\frac{^nC_r}{^nC_{r-1}} = \frac{n-r+1}{r}$

(iv) $^nC_r = \frac{n}{r}\,{}^{n-1}C_{r-1} = \frac{n(n-1)}{r(r-1)}\,{}^{n-2}C_{r-2} = ... = \frac{n(n-1)(n-2).........(n-(r-1))}{r\;(r-1)(r-2).......2\;.1}$

(v) If n and r are relatively prime, then ^nC_r is divisible by n. But converse is not necessarily true.

Statement of binomial theorem :

$(a + b)^n = {}^nC_0\, a^nb^0 + {}^nC_1\, a^{n-1}b^1 + {}^nC_2\, a^{n-2}b^2 + ... + {}^nC_r\, a^{n-r}b^r + ... + {}^nC_n\, a^0 b^n$

where $n \in N$

or $(a + b)^n = \sum_{r=0}^{n} {}^nC_r\; a^{n-r}b^r$

Note : If we put a = 1 and b = x in the above binomial expansion, then

or $(1 + x)^n = {}^nC_0 + {}^nC_1\, x + {}^nC_2\, x^2 + ... + {}^nC_r\, x^r + ... + {}^nC_n\, x^n$

or $(1 + x)^n = \sum_{r=0}^{n} {}^nC_r\; x^r$

Example # 1 : Expand the following binomials :

(i) $(x+\sqrt{2})^5$ (ii) $\left(1-\dfrac{3x^2}{2}\right)^4$

Solution : (i) $(x+\sqrt{2})^5 = {}^5C_0x^5 + {}^5C_1x^4\left(\sqrt{2}\right) + {}^5C_2x^3\left(\sqrt{2}\right)^2 + {}^5C_3\,x^2\left(\sqrt{2}\right)^3 + {}^5C_4x\left(\sqrt{2}\right)^4 + {}^5C_5\left(\sqrt{2}\right)^5$

$= x^5 + 5\sqrt{2}\,x^4 + 20x^3 + 20\sqrt{2}\,x^2 + 20x + 4\sqrt{2}$

(ii) $\left(1-\dfrac{3x^2}{2}\right)^4 = {}^4C_0 + {}^4C_1\left(-\dfrac{3x^2}{2}\right) + {}^4C_2\left(-\dfrac{3x^2}{2}\right)^2 + {}^4C_3\left(-\dfrac{3x^2}{2}\right)^3 + {}^4C_4\left(-\dfrac{3x^2}{2}\right)^4$

$= 1 - 6x^2\ \dfrac{27}{2} + x^4 - \dfrac{27}{2}\,x^6 + \dfrac{81}{16}\,x^8$

Example # 2 : Expand the binomial $\left(\dfrac{2}{x}+x\right)^{10}$ up to four terms

Solution : $\left(\dfrac{2}{x}+x\right)^{10} = {}^{10}C_0\left(\dfrac{2}{x}\right)^{10} + {}^{10}C_1\left(\dfrac{2}{x}\right)^9 x + {}^{10}C_2\left(\dfrac{2}{x}\right)^8 x^2 + {}^{10}C_3\left(\dfrac{2}{x}\right)^7 x^3 + ...$

Self practice problems :

(1) Write the first three terms in the expansion of $\left(2-\dfrac{y}{3}\right)^6$.

(2) Expand the binomial $\left(\dfrac{x^2}{3}+\dfrac{3}{x}\right)^5$.

Ans. (1) $64 - 64y + \dfrac{80}{3}\,y^2$ (2) $\dfrac{x^{10}}{243} + \dfrac{5}{27}\,x^7 + \dfrac{10}{3}x^4 + 30x + \dfrac{135}{x^2} + \dfrac{243}{x^5}$.

Observations :

(i) The number of terms in the binomial expansion $(a + b)^n$ is $n + 1$.

(ii) The sum of the indices of a and b in each term is n.

(iii) The binomial coefficients (${}^nC_0$, ${}^nC_1$ ....${}^nC_n$) of the terms equidistant from the beginning and the end are equal, i.e. ${}^nC_0 = {}^nC_n$, ${}^nC_1 = {}^nC_{n-1}$ etc. $\{\because\ {}^nC_r = {}^nC_{n-r}\}$

(iv) The binomial coefficient can be remembered with the help of the following pascal's Triangle (also known as Meru Prastra provided by Pingla)

Index of the binomial	The binomial coefficient
0	1
1	1 1
2	1 2 1
3	1 3 3 1
4	1 4 6 4 1
5	1 5 10 10 5 1

Regarding Pascal's Triangle, we note the following :

(a) Each row of the triangle begins with 1 and ends with 1.

(b) Any entry in a row is the sum of two entries in the preceding row, one on the immediate left and the other on the immediate right.

Example # 3 : The number of dissimilar terms in the expansion of $(1 + x^4 - 2x^2)^{15}$ is

(A) 21 (B) 31 (C) 41 (D) 61

Solution : $(1 - x^2)^{30}$

Therefore number of dissimilar terms = **31.**

General term :

$(x + y)^n = {}^nC_0\, x^n y^0 + {}^nC_1\, x^{n-1} y^1 + \dots\dots\dots + {}^nC_r\, x^{n-r} y^r + \dots\dots\dots + {}^nC_n\, x^0 y^n$

$(r + 1)^{th}$ term is called general term and denoted by T_{r+1}.

$T_{r+1} = {}^nC_r\, x^{n-r} y^r$

Note : The r^{th} term from the end is equal to the $(n - r + 2)^{th}$ term from the begining, i.e. ${}^nC_{n-r+1}\, x^{r-1} y^{n-r+1}$

Example # 4 : Find (i) 15^{th} term of $(2x - 3y)^{20}$ (ii) 4^{th} term of $\left(\frac{3x}{5} - y\right)^7$

Solution : (i) $T_{14+1} = {}^{20}C_{14}\,(2x)^6\,(-3y)^{14} = {}^{20}C_{14}\, 2^6\, 3^{14}\, x^6.y^{14}$

(ii) $T_{3+1} = {}^7C_3 \left(\frac{3x}{5}\right)^4 (-y)^3 = {}^{-7}C_3 \left(\frac{3}{5}\right)^4 x^4y^3$

Example # 5 : Find the number of rational terms in the expansion of $\left(2^{\frac{1}{3}} + 3^{\frac{1}{5}}\right)^{600}$

Solution : The general term in the expansion of $\left(2^{\frac{1}{3}} + 3^{\frac{1}{5}}\right)^{600}$ is

$T_{r+1} = {}^{600}C_r \left(2^{\frac{1}{3}}\right)^{600-r} \left(3^{\frac{1}{5}}\right)^r = {}^{600}C_r\ 2^{\frac{600-r}{3}}\ 3^{\frac{r}{5}}$

The above term will be rational if exponent of 3 and 2 are integers

It means $\frac{600-r}{3}$ and $\frac{r}{5}$ must be integers.

The possible set of values of r is {0, 15, 30,45............,600}

Hence, number of rational terms is 41

Middle term(s) :

(a) If n is even, there is only one middle term, which is $\left(\frac{n+2}{2}\right)^{th}$ term.

(b) If n is odd, there are two middle terms, which are $\left(\frac{n+1}{2}\right)^{th}$ and $\left(\frac{n+1}{2}+1\right)^{th}$ terms.

Example # 6 : Find the middle term(s) in the expansion of

(i) $(1 + 2x)^{12}$ (ii) $\left(2y - \frac{y^2}{2}\right)^{11}$

Solution : (i) $(1 + 2x)^{12}$

Here, n is even, therefore middle term is $\left(\frac{12+2}{2}\right)^{th}$ term.

It means T_7 is middle term $T_7 = {}^{12}C_6\,(2x)^6$

(ii) $\left(2y - \frac{y^2}{2}\right)^{11}$

Here, n is odd therefore, middle terms are $\left(\frac{11+1}{2}\right)^{th}$ & $\left(\frac{11+1}{2}+1\right)^{th}$.

It means T_6 & T_7 are middle terms

$T_6 = {}^{11}C_5\,(2y)^6 \left(-\frac{y^2}{2}\right)^5 = -2\ {}^{11}C_5\, y^{16} \Rightarrow T_7 = {}^{11}C_6\,(2y)^5 \left(-\frac{y^2}{2}\right)^6 = \frac{{}^{11}C_6}{2} y^{17}$

Example # 7 : Find term which is independent of x in $\left(x^2 - \frac{1}{x^6}\right)^{16}$

Solution : $T_{r+1} = {}^{16}C_r\,(x^2)^{16-r}\left(-\frac{1}{x^6}\right)^r$

For term to be independent of x, exponent of x should be 0

$32 - 2r = 6r \quad \Rightarrow \quad r = 4 \quad \therefore\ T_5$ is independent of x.

Numerically greatest term in the expansion of $(a + b)^n$, $n \in N$

Binomial expansion of $(a + b)^n$ is as follows **: –**

$(a + b)^n = {}^nC_0\, a^n b^0 + {}^nC_1\, a^{n-1} b^1 + {}^nC_2\, a^{n-2} b^2 + ... + {}^nC_r\, a^{n-r} b^r + ... + {}^nC_n\, a^0 b^n$

If we put certain values of a and b in RHS, then each term of binomial expansion will have certain value. The term having numerically greatest value is said to be numerically greatest term.

Let T_r and T_{r+1} be the r^{th} and $(r + 1)^{th}$ terms respectively

$T_r = {}^nC_{r-1}\ a^{n-(r-1)}\, b^{r-1}$

$T_{r+1} = {}^nC_r\, a^{n-r}\, b^r$

Now, $\left|\frac{T_{r+1}}{T_r}\right| = \left|\frac{{}^nC_r\ \ a^{n-r}\ \ b^r}{{}^nC_{r-1}\ a^{n-r+1} b^{r-1}}\right| = \frac{n-r+1}{r} \cdot \left|\frac{b}{a}\right|$

Consider $\left|\frac{T_{r+1}}{T_r}\right| \geq 1$

$\left(\frac{n-r+1}{r}\right)\left|\frac{b}{a}\right| \geq 1 \quad \Rightarrow \quad \frac{n+1}{r} - 1 \geq \left|\frac{a}{b}\right| \quad \Rightarrow r \leq \frac{n+1}{1+\left|\frac{a}{b}\right|}$

Case - I When $\frac{n+1}{1+\left|\frac{a}{b}\right|}$ is an integer (say m), then

(i) $T_{r+1} > T_r$ when $r < m$ $(r = 1, 2, 3 ...., m – 1)$

i.e. $T_2 > T_1,\ T_3 > T_2,\ ....,\ T_m > T_{m-1}$

(ii) $T_{r+1} = T_r$ when $r = m$

i.e. $T_{m+1} = T_m$

(iii) $T_{r+1} < T_r$ when $r > m$ $(r = m + 1, m + 2, ...n)$

i.e. $T_{m+2} < T_{m+1}\ ,\ T_{m+3} < T_{m+2}\ ,\ ....T_{n+1} < T_n$

Conclusion :

When $\frac{n+1}{1+\left|\frac{a}{b}\right|}$ is an integer, say m, then T_m and T_{m+1} will be numerically greatest terms (both terms are equal in magnitude)

Case - II

When is not an integer (Let its integral part be m), then

(i) $T_{r+1} > T_r$ when $r <$ $(r = 1, 2, 3,........, m–1, m)$

i.e. $T_2 > T_1\ ,\ T_3 > T_2,\ ....,\ T_{m+1} > T_m$

(ii) $T_{r+1} < T_r$ when $r >$ $(r = m + 1, m + 2, ...n)$

i.e. $T_{m+2} < T_{m+1}\ ,\ T_{m+3} < T_{m+2}\ ,\ ....,\ T_{n+1} < T_n$

Conclusion :

When is not an integer and its integral part is m, then T_{m+1} will be the numerically greatest term.

Note : (i) In any binomial expansion, the middle term(s) has greatest binomial coefficient.

In the expansion of $(a + b)^n$

If n	No. of greatest binomial coefficient	Greatest binomial coefficient
Even	1	$^nC_{n/2}$
Odd	2	$^nC_{(n-1)/2}$ and $^nC_{(n+1)/2}$

(Values of both these coefficients are equal)

(ii) In order to obtain the term having numerically greatest coefficient, put a = b = 1, and proceed as discussed above.

Example # 8 : Find the numerically greatest term in the expansion of $(7 - 3x)^{25}$ when $x = \frac{1}{3}$.

Solution : $m = \dfrac{n+1}{1+\left|\frac{a}{b}\right|} = \dfrac{25+1}{1+\left|\frac{7}{-1}\right|} = \dfrac{26}{8}$

[m] = 3 ([m] denotes GIF)

$\therefore$ T_4 is numerically greatest term

Self practice problems :

(3) Find the term independent of x in $\left(x^2 - \frac{3}{x}\right)^9$

(4) The sum of all rational terms in the expansion of $(3^{1/7} + 5^{1/2})^{14}$ is

(A) 3^2 (B) $3^2 + 5^7$ (C) $3^7 + 5^2$ (D) 5^7

(5) Find the coefficient of x^{-2} in $(1 + x^2 + x^4)\left(1-\frac{1}{x^2}\right)^{18}$

(6) Find the middle term(s) in the expansion of $(1 + 3x + 3x^2 + x^3)^{2n}$

(7) Find the numerically greatest term in the expansion of $(2 + 5x)^{21}$ when $x = \frac{2}{5}$.

Ans. (3) 28.3^7 (4) B (5) –681

(6) $^{6n}C_{3n} \cdot x^{3n}$ (7) $T_{11} = T_{12} = {}^{21}C_{10}\ 2^{21}$

Example # 9 : Show that $7^n + 5$ is divisible by 6, where n is a positive integer.

Solution : $7^n + 5 = (1 + 6)^n + 5 = {}^nC_0 + {}^nC_1 . 6 + {}^nC_2 . 6^2 + ... + {}^nC_n 6^n + 5.$

$= 6. C_1 + 6^2. C_2 + ... + C_n . 6^n + 6.$

$= 6\lambda$, where λ is a positive integer

Hence, $7^n + 5$ is divisible by 6.

Example # 10 : What is the remainder when 7^{81} is divided by 5.

Solution : $7^{81} = 7.7^{80} = 7. (49)^{40} = 7 (50 - 1)^{40}$

$= 7 [{}^{40}C_0 (50)^{40} - {}^{40}C_1 (50)^{39} + ... - {}^{40}C_{39} (50)^1 + {}^{40}C_{40} (50)^0]$

= 5(k) + 7(where k is a positive integer) = 5 (k + 1) + 2

Hence, remainder is 2.

Example # 11 : Find the last digit of the number $(13)^{12}$.

Solution : $(13)^{12} = (169)^6 = (170 - 1)^6$

$= {}^6C_0\,(170)^6 - {}^6C_1\,(170)^5 + \ldots\ldots - {}^6C_5\,(170)^1 + {}^6C_6\,(170)^0$

Hence, last digit is 1

Note : We can also conclude that last three digits are 481.

Example-12 : Which number is larger $(1.1)^{100000}$ or 10,000 ?

Solution : By Binomial Theorem

$(1.1)^{100000} = (1 + 0.1)^{100000} = 1 + {}^{100000}C_1\,(0.1)$ + other positive terms

$= 1 + 100000 \times 0.1$ + other positive terms

$= 1 + 10000$ + other positive terms

Hence $(1.1)^{100000} > 10,000$

Self practice problems :

(8) If n is a positive integer, then show that $6^n - 5n - 1$ is divisible by 25.

(9) What is the remainder when 3^{257} is divided by 80 .

(10) Find the last digit, last two digits and last three digits of the number $(81)^{25}$.

(11) Which number is larger $(1.3)^{2000}$ or 600

Ans. (9) 3 (10) 1, 01, 001 (11) $(1.3)^{2000}$.

Some standard expansions :

(i) Consider the expansion

$$(x + y)^n = \sum_{r=0}^{n} {}^nC_r\; x^{n-r}\, y^r = {}^nC_0\, x^n\, y^0 + {}^nC_1\, x^{n-1}\, y^1 + \ldots\ldots\ldots\ldots + {}^nC_r\, x^{n-r}\, y^r + \ldots\ldots\ldots\ldots + {}^nC_n\, x^0\, y^n \quad \ldots.(i)$$

(ii) Now replace $y \to -y$ we get

$$(x - y)^n = \sum_{r=0}^{n} {}^nC_r\; (-1)^r\, x^{n-r}\, y^r = {}^nC_0\, x^n\, y^0 - {}^nC_1\, x^{n-1}\, y^1 + \ldots + {}^nC_r\, (-1)^r x^{n-r}\, y^r + \ldots + {}^nC_n\, (-1)^n\, x^0\, y^n \quad \ldots.(ii)$$

(iii) Adding (i) & (ii), we get

$(x + y)^n + (x - y)^n = 2[{}^nC_0\, x^n\, y^0 + {}^nC_2\, x^{n-2}\, y^2 + \ldots\ldots\ldots]$

(iv) Subtracting (ii) from (i), we get

$(x + y)^n - (x - y)^n = 2[{}^nC_1\, x^{n-1}\, y^1 + {}^nC_3\, x^{n-3}\, y^3 + \ldots\ldots\ldots]$

Properties of binomial coefficients :

$(1 + x)^n = C_0 + C_1x + C_2x^2 + \ldots\ldots\ldots + C_rx^r + \ldots\ldots\ldots\ldots + C_nx^n$ ......(1)

where C_r denotes ${}^nC_r$

(1) The sum of the binomial coefficients in the expansion of $(1 + x)^n$ is 2^n

Putting x = 1 in (1)

$${}^nC_0 + {}^nC_1 + {}^nC_2 + \ldots\ldots\ldots + {}^nC_n = 2^n \quad \ldots\ldots(2)$$

or $\displaystyle\sum_{r=0}^{n} {}^nC_r = 2^n$

(2) Again putting x = –1 in (1), we get

$${}^nC_0 - {}^nC_1 + {}^nC_2 - {}^nC_3 + \ldots\ldots\ldots\ldots + (-1)^n\, {}^nC_n = 0 \quad \ldots\ldots(3)$$

or $\displaystyle\sum_{r=0}^{n} (-1)^r\; {}^nC_r = 0$

(3) The sum of the binomial coefficients at odd position is equal to the sum of the binomial coefficients at even position and each is equal to 2^{n-1}.
from (2) and (3)

$${}^nC_0 + {}^nC_2 + {}^nC_4 + \ldots\ldots\ldots\ldots\ldots = {}^nC_1 + {}^nC_3 + {}^nC_5 + \ldots\ldots\ldots\ldots\ldots = 2^{n-1}$$

(4) Sum of two consecutive binomial coefficients
$${}^nC_r + {}^nC_{r-1} = {}^{n+1}C_r$$

L.H.S. $= {}^nC_r + {}^nC_{r-1} = \dfrac{n!}{(n-r)!\ r!} + \dfrac{n!}{(n-r+1)!\ (r-1)!}$

$$= \frac{n!}{(n-r)!\ (r-1)!}\left[\frac{1}{r} + \frac{1}{n-r+1}\right]$$

$$= \frac{n!}{(n-r)!\ (r-1)!}\ \frac{(n+1)}{r(n-r+1)}$$

$$= \frac{(n+1)!}{(n-r+1)!\ r!} = {}^{n+1}C_r = \text{R.H.S.}$$

(5) Ratio of two consecutive binomial coefficients
$$\frac{{}^nC_r}{{}^nC_{r-1}} = \frac{n-r+1}{r}$$

(6) $${}^nC_r = \frac{n}{r}\ {}^{n-1}C_{r-1} = \frac{n(n-1)}{r(r-1)}\ {}^{n-2}C_{r-2} = \ldots\ldots\ldots\ldots = \frac{n(n-1)(n-2)\ldots\ldots\ldots(n-(r-1))}{r\ (r-1)(r-2)\ldots\ldots.2\ .1}$$

Example # 13 : If $(1 + x)^n = C_0 + C_1x + C_2x^2 + \ldots\ldots\ldots\ldots + c_nx^n$, then show that
(i) $C_0 + 4C_1 + 4^2C_2 + \ldots\ldots\ldots. + 4^n C_n = 5^n$. (ii) $3C_0 + 5C_1 + 7. C_2 + \ldots\ldots.. + (2n + 3) C_n = 2^n (n + 3)$.

(iii) $$C_0 + \frac{C_1}{2} + \frac{C_2}{3} + \frac{C_3}{4} + \ldots\ldots\ldots + \frac{C_n}{n+1} = \frac{2^{n+1}-1}{n+1}$$

Solution : (i) $(1 + x)^n = C_0 + C_1 x + C_2x^2 + \ldots\ldots\ldots.. + C_nx^n$
put x = 4
$C_0 + 4C_1 + 4^2C_2 + \ldots\ldots\ldots. + 4^n C_n = 5^n$.

(ii) L.H.S. $= 3C_0 + 5C_1 + 7. C_2 + \ldots\ldots.. + (2n + 3) C_n$

$$= \sum_{r=0}^{n}(2r+3)\ .\ {}^nC_r = 2\sum_{r=0}^{n} r.\ {}^nC_r + 3\sum_{r=0}^{n}{}^nC_r$$

$$= 2n\sum_{r=1}^{n}{}^{n-1}C_{r-1} + 3\sum_{r=0}^{n}{}^nC_r = 2n\ .\ 2^{n-1} + 3.\ 2^n = 2^n (n + 3)\ \text{RHS}$$

(iii) I **<u>Method : By Summation</u>**

L.H.S. $= C_0 + \dfrac{C_1}{2} + \dfrac{C_2}{3} + \dfrac{C_3}{4} + \ldots\ldots\ldots + \dfrac{C_n}{n+1}$

$$= \sum_{r=0}^{n} \frac{{}^nC_r}{r+1} = \frac{1}{n+1}\sum_{r=0}^{n}\ .\ {}^{n+1}C_{r+1} \qquad \left\{\frac{n+1}{r+1}\ .\ {}^nC_r = {}^{n+1}C_{r+1}\right\} = \frac{2^{n+1}-1}{n+1}\ \text{R.H.S.}$$

II **<u>Method : By Integration</u>**

$(1 + x)^n = C_0 + C_1x + C_2x^2 + \ldots\ldots + C_n x^n$. Integrating both sides, within the limits 0 to 1.

$$\left[\frac{(1+x)^{n+1}}{n+1}\right]_0^1 = \left[C_0x + C_1\frac{x^2}{2} + C_2\frac{x^3}{3} + \ldots.. + C_n\frac{x^{n+1}}{n+1}\right]_0^1$$

$$\frac{2^{n+1}}{n+1} - \frac{1}{n+1} = \left(C_0 + \frac{C_1}{2} + \frac{C_2}{3} + \ldots.. + \frac{C_n}{n+1}\right) - 0$$

$$C_0 + \frac{C_1}{2} + \frac{C_2}{3} + \frac{C_3}{4} + \ldots\ldots\ldots + \frac{C_n}{n+1} = \frac{2^{n+1}-1}{n+1}\ \text{Proved}$$

Example # 14 : If $(1 + x)^n = C_0 + C_1x + C_2x^2 + ...+ C_nx^n$, then prove that

(i) $C_0C_1 + C_1C_2 + C_2C_3 + ... + C_{n-1} C_n = {}^{2n}C_{n-1}$ or ${}^{2n}C_{n+1}$

(ii) $1^2. C_1^2 + 2^2 . C_2^2 + 3^2. C_3^2 + ... + n^2 C_n^2 . = n^2. {}^{2n-2}C_{n-1}$

Solution : (i) $(1 + x)^n = C_0 + C_1x + C_2x^2 + ... + C_n x^n$. ....(i)

$(x + 1)^n = C_0x^n + C_1x^{n-1} + C_2x^{n-2} + ... + C_n x^0$ ....(ii)

Multiplying (i) and (ii)

$(C_0 + C_1x + C_2x^2 + ... + C_nx^n) (C_0x^n + C_1x^{n-1} + ... + C_nx^0) = (1 + x)^{2n}$

Comparing coefficient of x^{n-1},

$C_0C_1 + C_1C_2 + C_2C_3 + ... + C_{n-1} C_n = {}^{2n}C_{n-1}$ or ${}^{2n}C_{n+1}$

(ii) $(1 + x)^n = C_0 + C_1x + C_2x^2 + ... + C_n x^n$. ....(i)

differentiating w.r.t x......

$n(1 + x)^{n-1} = C_1 + 2C_2x + 3C_3x^2 + ... + nC_n x^{n-1}$.

multiplying by x.......

$n x(1 + x)^{n-1} = C_1x + 2C_2x^2 + 3C_3x^3 + ... + nC_n x^n$

Now differentiate w.r.t. x.....

$n(1 + x)^{n-1} + n (n-1)x.(1+x)^{n-2} = 1^2C_1 + 2^2C_2x + 3^2C_3x^2 + ... + n^2C_nx^{n-1}$ ....(ii)

$(x + 1)^n = C_0x^n + C_1x^{n-1} + C_2x^{n-2} + ... + C_n x^0$ ....(iii)

multiplying (ii) & (iii) and comparing the cofficient of x^{n-1}

$1^2. C_1^2 + 2^2 . C_2^2 + 3^2. C_3^2 + ... + n^2 C_n^2. = n\left({}^{2n-1}C_{n-1} - {}^{2n-2}C_{n-2}\right) + n^2\, {}^{2n-2}C_{n-2}$

$= n^2\, {}^{2n-2}C_{n-1}$ = R.H.S.

Example # 15 : Find the summation of the following series –

(i) ${}^mC_0 + {}^{m+1}C_1 + {}^{m+2}C_2 + ... + {}^nC_m$ (ii) ${}^nC_3 + 2 . {}^{n+1}C_3 + 3. {}^{n+2}C_3 + ... + n . {}^{2n-1}C_3$

Solution : (i) I **Method :** Using property, ${}^nC_r + {}^nC_{r-1} = {}^{n+1}C_r$

${}^mC_0 + {}^{m+1}C_1 + {}^{m+2}C_2 + ... + {}^nC_m$

${}^mC_m + {}^{m+1}C_m + {}^{m+2}C_m + ... + {}^nC_m$

$= \underbrace{{}^{m+1}C_{m+1} + {}^{m+1}C_m} + {}^{m+2}C_m + ... + {}^nC_m$ $\{\because {}^mC_m = {}^{m+1}C_{m+1}\}$

$= \underbrace{{}^{m+2}C_{m+1} + {}^{m+2}C_m} + ... + {}^nC_m = {}^{m+3}C_{m+1} + ... + {}^nC_m = {}^nC_{m+1} + {}^nC_m = \mathbf{{}^{n+1}C_{m+1}}$

II **Method**

${}^mC_m + {}^{m+1}C_m + {}^{m+2}C_m + ... + {}^nC_m$

The above series can be obtained by writing the coefficient of x^m in

$(1 + x)^m + (1 + x)^{m+1} + ... + (1 + x)^n$

Let $S = (1 + x)^m + (1 + x)^{m+1} +............. + (1 + x)^n$

$$= \frac{(1+x)^m\left[(1+x)^{n-m+1} - 1\right]}{x} = \frac{(1+x)^{n+1} - (1+x)^m}{x}$$

$$= \text{coefficient of } x^m \text{ in } \frac{(1+x)^{n+1}}{x} - \frac{(1+x)^m}{x} = {}^{n+1}C_{m+1} + 0 = {}^{n+1}C_{m+1}$$

(ii) $^nC_3 + 2 \,.\, ^{n+1}C_3 + 3 \,.\, ^{n+2}C_3 + ... + n \,.\, ^{2n-1}C_3$

The above series can be obatined by writing the coefficient of x^3 in

$(1 + x)^n + 2 \,.\, (1 + x)^{n+1} + 3 \,.\, (1 + x)^{n+2} + ... + n \,.\, (1 + x)^{2n-1}$

Let $S = (1 + x)^n + 2 \,.\, (1 + x)^{n+1} + 3.\, (1 + x)^{n+2} + ... + n\, (1 + x)^{2n-1}$ .....(i)

$(1 + x)S = (1 + x)^{n+1} + 2\, (1 + x)^{n+2} + ... + (n - 1)\, (1 + x)^{2n-1} + n(1 + x)^{2n}$ ....(ii)

Subtracting (ii) from (i)

$- xS = (1 + x)^n + (1 + x)^{n+1} + (1 + x)^{n+2} + ... + (1 + x)^{2n-1} - n(1 + x)^{2n}$

$= \dfrac{(1+x)^n\left[(1+x)^n - 1\right]}{x} - n\,(1 + x)^{2n}$

$S = \dfrac{-(1+x)^{2n} + (1+x)^n}{x^2} + \dfrac{n(1+x)^{2n}}{x}$

$x^3 : S$ (coefficient of x^3 in S)

$x^3 : \dfrac{-(1+x)^{2n} + (1+x)^n}{x^2} + \dfrac{n(1+x)^{2n}}{x}$

Hence, required summation of the series is $-\,^{2n}C_5 + {}^nC_5 + n \,.\, ^{2n}C_4$

Example # 16 : Prove that $C_1 - C_3 + C_5 - .... = 2^{n/2} \sin\dfrac{n\pi}{4}$.

Solution : Consider the expansion $(1 + x)^n = C_0 + C_1 x + C_2 x^2 + ... + C_n x^n$ ....(i)

putting $x = - i$ in (i) we get

$(1 - i)^n = C_0 - C_1 i - C_2 + C_3 i + C_4 + ...\ (-1)^n C_n i^n$

or $2^{n/2}\left[\cos\left(-\dfrac{n\pi}{4}\right) + i \sin\left(-\dfrac{n\pi}{4}\right)\right] = (C_0 - C_2 + C_4 - ....) - i\,(C_1 - C_3 + C_5 - .....)$ ....(ii)

Equating the imaginary part in (ii) we get $C_1 - C_3 + C_5 - .... = 2^{n/2} \sin\dfrac{n\pi}{4}$.

Self practice problems :

(12) Prove the following

(i) $5C_0 + 7C_1 + 9C_2 + ... + (2n + 5)\, C_n = 2^n\,(n + 5)$

(ii) $4C_0 + \dfrac{4^2}{2} \,.\, C_1 + \dfrac{4^3}{3} C_2 + ... + \dfrac{4^{n+1}}{n+1} C_n = \dfrac{5^{n+1} - 1}{n+1}$

(iii) $^nC_0 \,.\, ^{n+1}C_n + {}^nC_1 \,.\, ^nC_{n-1} + {}^nC_2 \,.\, ^{n-1}C_{n-2} + ... + {}^nC_n \,.\, ^1C_0 = 2^{n-1}\,(n + 2)$

(iv) $^2C_2 + {}^3C_2 + ... + {}^nC_2 = {}^{n+1}C_3$

Binomial theorem for negative and fractional indices :

If $n \in R$, then $(1 + x)^n = 1 + nx + \dfrac{n(n-1)}{2\,!} x^2 + \dfrac{n(n-1)(n-2)}{3\,!} x^3 + ...$

$.................. + \dfrac{n(n-1)(n-2).......(n-r+1)}{r\,!} x^r + ... \infty.$

Remarks

(i) The above expansion is valid for any rational number other than a whole number if $|\, x \,| < 1$.

(ii) When the index is a negative integer or a fraction then number of terms in the expansion of $(1 + x)^n$ is infinite, and the symbol ^nC_r cannot be used to denote the coefficient of the general term.

(iii) The first term must be unity in the expansion, when index 'n' is a negative integer or fraction

$$(x+y)^n = \begin{cases} x^n\left(1+\dfrac{y}{x}\right)^n = x^n\left\{1+n\ .\ \dfrac{y}{x}+\dfrac{n\ (n-1)}{2\ !}\left(\dfrac{y}{x}\right)^2+.....\right\} & \text{if } \left|\dfrac{y}{x}\right|<1 \\ y^n\left(1+\dfrac{x}{y}\right)^n = y^n\left\{1+n\ .\ \dfrac{x}{y}+\dfrac{n\ (n-1)}{2\ !}\left(\dfrac{x}{y}\right)^2+.....\right\} & \text{if } \left|\dfrac{x}{y}\right|<1 \end{cases}$$

(iv) The general term in the expansion of $(1 + x)^n$ is $T_{r+1} = \dfrac{n(n-1)(n-2).........(n-r+1)}{r\ !}\ x^r$

(v) When 'n' is any rational number other than whole number then approximate value of $(1 + x)^n$ is $1 + nx$ (x^2 and higher powers of x can be neglected)

(vi) Expansions to be remembered ($|x| < 1$)

(a) $(1 + x)^{-1} = 1 - x + x^2 - x^3 + ... + (-1)^r x^r + ...\infty$

(b) $(1 - x)^{-1} = 1 + x + x^2 + x^3 + ... + x^r + ...\infty$

(c) $(1 + x)^{-2} = 1 - 2x + 3x^2 - 4x^3 + ... + (-1)^r (r + 1) x^r + ...\infty$

(d) $(1 - x)^{-2} = 1 + 2x + 3x^2 + 4x^3 + ... + (r + 1)x^r + ... \infty$

Example # 17 : Prove that the coefficient of x^r in $(1 - x)^{-n}$ is ${}^{n+r-1}C_r$

Solution: $(r + 1)^{th}$ term in the expansion of $(1 - x)^{-n}$ can be written as

$$T_{r+1} = \frac{-n(-n-1)(-n-2)......(-n-r+1)}{r\ !}\ (-x)^r$$

$$= (-1)^r\ \frac{n(n+1)(n+2)......(n+r-1)}{r\ !}\ (-x)^r = \frac{n(n+1)(n+2)......(n+r-1)}{r\ !}\ x^r$$

$$= \frac{(n-1)!\ \ n(n+1)......(n+r-1)}{(n-1)\ !\ r\ !}\ x^r$$ Hence, coefficient of x^r is $\dfrac{(n+r-1)!}{(n-1)!\ \ r!} = {}^{n+r-1}C_r$ Proved

Example-18 : If x is so small such that its square and higher powers may be neglected, then find the value of $\dfrac{(1-2x)^{1/3}+(1+5x)^{-3/2}}{(9+x)^{1/2}}$

Solution : $$\frac{(1-2x)^{1/3}+(1+5x)^{-3/2}}{(9+x)^{1/2}} = \frac{1-\dfrac{2}{3}x+1-\dfrac{15x}{2}}{3\left(1+\dfrac{x}{9}\right)^{1/2}} = \frac{1}{3}\left(2-\frac{49}{6}x\right)\left(1+\frac{x}{9}\right)^{-1/2}$$

$$= \frac{1}{3}\left(2-\frac{49}{6}x\right)\left(1-\frac{x}{18}\right) = \frac{1}{2}\left(2-\frac{x}{9}-\frac{49}{6}x\right) = 1-\frac{x}{18}-\frac{49}{12}x = 1-\frac{149}{36}x$$

Self practice problems :

(13) Find the possible set of values of x for which expansion of $(3 - 2x)^{1/2}$ is valid in ascending powers of x.

(14) If $y = \dfrac{2}{5} + \dfrac{1.3}{2\ !}\left(\dfrac{2}{5}\right)^2 + \dfrac{1.3.5}{3\ !}\left(\dfrac{2}{5}\right)^3 + ...$, then find the value of $y^2 + 2y$

(15) The coefficient of x^{50} in $\dfrac{2-3x}{(1-x)^3}$ is

(A) 500 (B) 1000 (C) –1173 (D) 1173

Ans. (13) $x \in \left(-\dfrac{3}{2}, \dfrac{3}{2}\right)$ (14) 4 (15) C

Multinomial theorem : As we know the Binomial Theorem $(x + y)^n = \sum_{r=0}^{n} {}^nC_r \; x^{n-r} y^r = \sum_{r=0}^{n} \frac{n!}{(n-r)!\; r!} x^{n-r} y^r$

putting $n - r = r_1$, $r = r_2$ therefore, $(x + y)^n = \sum_{r_1+r_2=n} \frac{n!}{r_1!\; r_2!} x^{r_1} \,.\, y^{r_2}$

Total number of terms in the expansion of $(x + y)^n$ is equal to number of non-negative integral solution of $r_1 + r_2 = n$ i.e. ${}^{n+2-1}C_{2-1} = {}^{n+1}C_1 = n + 1$

In the same fashion we can write the multinomial theorem

$$(x_1 + x_2 + x_3 + ... x_k)^n = \sum_{r_1+r_2+...+r_k=n} \frac{n!}{r_1!\; r_2!...\; r_k!} x_1^{r_1} \,.\, x_2^{r_2}...x_k^{r_k}$$

Here total number of terms in the expansion of $(x_1 + x_2 + ... + x_k)^n$ is equal to number of non-negative integral solution of $r_1 + r_2 + ... + r_k = n$ i.e. ${}^{n+k-1}C_{k-1}$

Example # 19 : Find the coefficient of $a^2 b^3 c^4 d$ in the expansion of $(a - b - c + d)^{10}$

Solution : $(a - b - c + d)^{10} = \sum_{r_1+r_2+r_3+r_4=10} \frac{(10)!}{r_1!\; r_2! r_3!\; r_4!} (a)^{r_1} \; (-b)^{r_2} \; (-c)^{r_3} \; (d)^{r_4}$

we want to get $a^2 b^3 c^4 d$ this implies that $r_1 = 2, r_2 = 3, r_3 = 4, r_4 = 1$

$\therefore$ coeff. of $a^2 b^3 c^4 d$ is $\frac{(10)!}{2!\; 3!\; 4!\; 1!} (-1)^3 (-1)^4 = -12600$

Example # 20 : In the expansion of $\left(1 + x + \frac{7}{x}\right)^{11}$, find the term independent of x.

Solution : $\left(1 + x + \frac{7}{x}\right)^{11} = \sum_{r_1+r_2+r_3=11} \frac{(11)!}{r_1!\; r_2!\; r_3!} (1)^{r_1} \; (x)^{r_2} \; \left(\frac{7}{x}\right)^{r_3}$

The exponent 11 is to be divided among the base variables 1, x and $\frac{7}{x}$ in such a way so that we get x^0. Therefore, possible set of values of (r_1, r_2, r_3) are (11, 0, 0), (9, 1, 1), (7, 2, 2), (5, 3, 3), (3, 4, 4), (1, 5, 5)

Hence the required term is

$$\frac{(11)!}{(11)!} (7^0) + \frac{(11)!}{9!\; 1\,!1\,!} 7^1 + \frac{(11)!}{7!\; 2\,!\; 2\,!} 7^2 + \frac{(11)!}{5!\; 3\,!\; 3\,!} 7^3 + \frac{(11)!}{3!\; 4\,!\; 4\,!} 7^4 + \frac{(11)!}{1\,!\; 5\,!\; 5\,!} 7^5$$

$$= 1 + \frac{(11)!}{9\,!\; 2\,!} \cdot \frac{2\,!}{1\,!\; 1\,!} 7^1 + \frac{(11)!}{7\,!\; 4\,!} \cdot \frac{4\,!}{2\,!\; 2\,!} 7^2 + \frac{(11)\,!}{5\,!\; 6\,!} \cdot \frac{6\,!}{3\,!\; 3\,!} 7^3$$

$$+ \frac{(11)\,!}{3\,!\; 8\,!} \cdot \frac{8\,!}{4\,!\; 4\,!} 7^4 + \frac{(11)\,!}{1\,!\; 10\,!} \cdot \frac{(10)\,!}{5\,!\; 5\,!} 7^5$$

$$= 1 + {}^{11}C_2 \,.\, {}^2C_1 \,.\, 7^1 + {}^{11}C_4 \,.\, {}^4C_2 \,.\, 7^2 + {}^{11}C_6 \,.\, {}^6C_3 \,.\, 7^3 + {}^{11}C_8 \,.\, {}^8C_4 \,.\, 7^4 + {}^{11}C_{10} \,.\, {}^{10}C_5 \,.\, 7^5 = 1 + \sum_{r=1}^{5} {}^{11}C_{2r} \,.\, {}^{2r}C_r \,.\, 7^r$$

Self practice problems :

(16) The number of terms in the expansion of $(a + b + c + d + e)^n$ is
(A) ${}^{n+4}C_4$ (B) ${}^{n+3}C_n$ (C) ${}^{n+5}C_n$ (D) n + 1

(17) Find the coefficient of $x^2 y^3 z^1$ in the expansion of $(x - 2y - 3z)^7$

(18) Find the coefficient of x^{17} in $(2x^2 - x - 3)^9$

Ans. (16) A (17) $\frac{7!}{2!\; 3!\; 1!}$ 24 (18) 2304

Exercise-1

✍ **Marked questions are recommended for Revision.**

PART - I : SUBJECTIVE QUESTIONS

Section (A) : General Term & Coefficient of x^k in $(ax + b)^n$

A-1. Expand the following :

(i) $\left(\frac{2}{x}-\frac{x}{2}\right)^5$, $(x \neq 0)$ (ii) $\left(y^2+\frac{2}{y}\right)^4$, $(y \neq 0)$

A-2. In the binomial expansion of $\left(\sqrt[3]{2}+\frac{1}{\sqrt[3]{3}}\right)^n$, the ratio of the 7th term from the begining to the 7th term from the end is 1 : 6 ; find n.

A-3. Find the degree of the polynomial $\left(x+(x^3-1)^{1/2}\right)^5+\left(x-(x^3-1)^{1/2}\right)^5$.

A-4. Find the coefficient of

(i) x^6y^3 in $(x + y)^9$ (ii) $a^5 b^7$ in $(a - 2b)^{12}$

A-5. Find the co-efficient of x^7 in $\left(ax^2+\frac{1}{bx}\right)^{11}$ and of x^{-7} in $\left(ax-\frac{1}{bx^2}\right)^{11}$ and find the relation between 'a' & 'b' so that these co-efficients are equal. (where a, b $\neq$ 0).

A-6. Find the term independent of 'x' in the expansion of the expression,

$(1 + x + 2x^3)\left(\frac{3}{2}x^2-\frac{1}{3x}\right)^9$.

A-7. (i) Find the coefficient of x^5 in $(1 + 2x)^6(1 - x)^7$.

(ii) Find the coefficient of x^4 in $(1 + 2x)^4 (2 - x)^5$

A-8.✍ In the expansion of $\left(x^3-\frac{1}{x^2}\right)^n$, $n \in N$, if the sum of the coefficients of x^5 and x^{10} is 0, then n is :

Section (B) : Middle term, Remainder & Numerically/Algebrically Greatest terms

B-1. Find the middle term(s) in the expansion of

(i) $\left(\frac{x}{y}-\frac{y}{x}\right)^7$ (ii) $(1 - 2x + x^2)^n$

B-2.✍ Prove that the co-efficient of the middle term in the expansion of $(1 + x)^{2n}$ is equal to the sum of the co-efficients of middle terms in the expansion of $(1 + x)^{2n-1}$.

B-3. (i) Find the remainder when 7^{98} is divided by 5

(ii) Using binomial theorem prove that $6^n - 5n$ always leaves the remainder 1 when divided by 25.

(iii) Find the last digit, last two digits and last three digits of the number $(27)^{27}$.

B-4.✍ Which is larger : $(99^{50} + 100^{50})$ or $(101)^{50}$.

B-5. (i) Find numerically greatest term(s) in the expansion of $(3 - 5x)^{15}$ when $x = \frac{1}{5}$

(ii) Which term is the numerically greatest term in the expansion of $(2x + 5y)^{34}$, when $x = 3$ & $y = 2$?

B-6. Find-

Section (D) : Negative & fractional index, Multinomial theorem

D-1. Find the co-efficient of x^6 in the expansion of $(1 - 2x)^{-5/2}$.

D-2. (i) Find the coefficient of x^{12} in $\frac{4+2x-x^2}{(1+x)^3}$

(ii) Find the coefficient of x^{100} in $\frac{3-5x}{(1-x)^2}$

D-3. Assuming ' x ' to be so small that x^2 and higher powers of ' x ' can be neglected, show that, $\frac{\left(1+\frac{3}{4}x\right)^{-4}(16-3x)^{1/2}}{(8+x)^{2/3}}$ is approximately equal to, $1 - \frac{305}{96}x$.

D-4. (i) Find the coefficient of $a^5 b^4 c^7$ in the expansion of $(bc + ca + ab)^8$.

(ii) Sum of coefficients of odd powers of x in expansion of $(9x^2 + x - 8)^6$

D-5. Find the coefficient of x^7 in $(1 - 2x + x^3)^5$.

PART - II : ONLY ONE OPTION CORRECT TYPE

Section (A) : General Term & Coefficient of x^k in $(ax +b)^n$

A-1. The $(m + 1)^{th}$ term of $\left(\frac{x}{y}+\frac{y}{x}\right)^{2m+1}$ is:

(A) independent of x
(B) a constant
(C) depends on the ratio x/y and m
(D) none of these

A-2. The total number of distinct terms in the expansion of, $(x + a)^{100} + (x - a)^{100}$ after simplification is :

(A) 50 (B) 202 (C) 51 (D) none of these

A-3. The value of, $\frac{18^3+7^3+3\,.\,18\,.\,7\,.\,25}{3^6+6.\,243.\,2+15\,.\,81\,.\,4+20.\,27.\,8+15.\,9\,.\,16+6.3.\,32+64}$ is :

(A) 1 (B) 2 (C) 3 (D) none

A-4. In the expansion of $\left(3-\sqrt{\frac{17}{4}+3\sqrt{2}}\right)^{15}$ the 11th term is a:

(A) positive integer
(B) positive irrational number
(C) negative integer
(D) negative irrational number.

A-5. If the second term of the expansion $\left[a^{1/13}+\frac{a}{\sqrt{a^{-1}}}\right]^n$ is $14a^{5/2}$, then the value of $\frac{{}^nC_3}{{}^nC_2}$ is:

(A) 4 (B) 3 (C) 12 (D) 6

A-6. In the expansion of $(7^{1/3} + 11^{1/9})^{6561}$, the number of terms free from radicals is:

(A) 730 (B) 729 (C) 725 (D) 750

A-7. The value of m, for which the coefficients of the $(2m + 1)^{th}$ and $(4m + 5)^{th}$ terms in the expansion of $(1 + x)^{10}$ are equal, is

(A) 3 (B) 1 (C) 5 (D) 8

A-8. The co-efficient of x in the expansion of $(1 - 2x^3 + 3x^5)\left(1+\frac{1}{x}\right)^8$ is :

(A) 56 (B) 65 (C) 154 (D) 62

A-9. Given that the term of the expansion $(x^{1/3} - x^{-1/2})^{15}$ which does not contain x is 5 m, where m $\in$ N,then m=

(A) 1100 (B) 1010 (C) 1001 (D) 1002

A-10. The term independent of x in the expansion of $\left(x - \frac{1}{x}\right)^4 \left(x + \frac{1}{x}\right)^3$ is:

(A) – 3 (B) 0 (C) 1 (D) 3

Section (B) : Middle term, Remainder & Numerically/Algebrically Greatest terms

B-1. If $k \in R^+$ and the middle term of $\left(\frac{k}{2}+2\right)^8$ is 1120, then value of k is:

(A) 3 (B) 2 (C) 1 (D) 4

B-2. The remainder when 2^{2003} is divided by 17 is :

(A) 1 (B) 2 (C) 8 (D) 7

B-3. The last two digits of the number 3^{400} are:

(A) 81 (B) 43 (C) 29 (D) 01

B-4. The last three digits in 10 ! are :

(A) 800 (B) 700 (C) 500 (D) 600

B-5. The value of $\sum_{r=1}^{10} r \cdot \frac{{}^nC_r}{{}^nC_{r-1}}$ is equal to

(A) 5 (2n – 9) (B) 10 n (C) 9 (n – 4) (D) n –2

B-6. $\sum_{r=0}^{n-1} \frac{{}^nC_r}{{}^nC_r + {}^nC_{r+1}} =$

(A) $\frac{n}{2}$ (B) $\frac{n+1}{2}$ (C) $(n+1)\ \frac{n}{2}$ (D) $\frac{n\ (n-1)}{2\ (n+1)}$

B-7. Find numerically greatest term in the expansion of $(2 + 3x)^9$, when x = 3/2.

(A) ${}^9C_6 . 2^9 . (3/2)^{12}$ (B) ${}^9C_3 . 2^9 . (3/2)^6$ (C) ${}^9C_5 . 2^9 . (3/2)^{10}$ (D) ${}^9C_4 . 2^9 . (3/2)^8$

B-8. The greatest integer less than or equal to $(\sqrt{2} + 1)^6$ is

(A) 196 (B) 197 (C) 198 (D) 199

Section (C) : Summation of series, Variable upper index & Product of binomial coefficients

C-1. $\frac{{}^{11}C_0}{1} + \frac{{}^{11}C_1}{2} + \frac{{}^{11}C_2}{3} + ... + \frac{{}^{11}C_{10}}{11} =$

(A) $\frac{2^{11}-1}{11}$ (B) $\frac{2^{11}-1}{6}$ (C) $\frac{3^{11}-1}{11}$ (D) $\frac{3^{11}-1}{6}$

C-2. The value of $\frac{C_0}{1.3} - \frac{C_1}{2.3} + \frac{C_2}{3.3} - \frac{C_3}{4.3} + ...+ (-1)^n \frac{C_n}{(n+1)\ .\ 3}$ is :

(A) $\frac{3}{n+1}$ (B) $\frac{n+1}{3}$ (C) $\frac{1}{3(n+1)}$ (D) none of these

C-3. The value of the expression ${}^{47}C_4 + \sum_{j=1}^{5} {}^{52-j}C_3$ is equal to :

(A) ${}^{47}C_5$ (B) ${}^{52}C_5$ (C) ${}^{52}C_4$ (D) ${}^{49}C_4$

C-4. The value of $\binom{50}{0}\binom{50}{1} + \binom{50}{1}\binom{50}{2} + ...+ \binom{50}{49}\binom{50}{50}$ is, where ${}^nC_r = \binom{n}{r}$

(A) $\binom{100}{50}$ (B) $\binom{100}{51}$ (C) $\binom{50}{25}$ (D) $\binom{50}{25}^2$

Section (D) : Negative & fractional index, Multinomial theorem

D-1. If $|x| < 1$, then the co-efficient of x^n in the expansion of $(1 + x + x^2 + x^3 +.......)^2$ is

(A) n (B) n – 1 (C) n + 2 (D) n + 1

D-2. The co–efficient of x^4 in the expansion of $(1 - x + 2x^2)^{12}$ is:

(A) ${}^{12}C_3$ (B) ${}^{13}C_3$ (C) ${}^{14}C_4$ (D) ${}^{12}C_3 + 3\ {}^{13}C_3 + {}^{14}C_4$

D-3. If $(1 + x)^{10} = a_0 + a_1x + a_2x^2 +......+ a_{10}x^{10}$, then value of

$(a_0 - a_2 + a_4 - a_6 + a_8 - a_{10})^2 + (a_1 - a_3 + a_5 - a_7 + a_9)^2$ is

(A) 2^{10} (B) 2 (C) 2^{20} (D) None of these

PART - III : MATCH THE COLUMN

1. **Column – I** **Column – II**

(A) If $(r + 1)^{th}$ term is the first negative term in the expansion of $(1 + x)^{7/2}$, then the value of r (where $0 < x < 1$) is — (p) divisible by 2

(B) If the sum of the co-efficients in the expansion of $(1 + 2x)^n$ is 6561, and T_r is the greatest term in the expansion for x = 1/2 then r is — (q) divisible by 5

(C) ${}^nC_r$ is divisible by n, $(1 < r < n)$ if n is — (r) divisible by 10

(D) The coefficient of x^4 in the expression $(1 + 2x + 3x^2 + 4x^3 + ...\text{up to } \infty)^{1/2}$ is c, $(c \in N)$, then c + 1 (where $|x| < 1$) is — (s) a prime number

Exercise-2

✍ **Marked questions are recommended for Revision.**

PART - I : ONLY ONE OPTION CORRECT TYPE

1. In the expansion of

$\left(3\sqrt{\frac{a}{b}}+3\sqrt{\frac{b}{\sqrt{a}}}\right)^{21}$, the term containing same powers of a & b is

(A) 11^{th} term (B) 13^{th} term (C) 12^{th} term (D) 6^{th} term

2. Consider the following statements :

S_1 : Number of dissimilar terms in the expansion of $(1 + x + x^2 + x^3)^n$ is $3n + 1$

S_2 : $(1 + x)(1 + x + x^2)(1 + x + x^2 + x^3)\ldots\ldots(1 + x + x^2 + \ldots\ldots + x^{100})$ when written in the ascending power of x then the highest exponent of x is 5000.

S_3 : $\sum_{k=1}^{n-r} {}^{n-k}C_r = {}^{n}C_{r+1}$

S_4 : If $(1 + x + x^2)^n = a_0 + a_1x + a_2x^2 + \ldots\ldots + a_{2n}x^{2n}$, then $a_0 + a_2 + a_4 + \ldots\ldots + a_{2n} = \frac{3^n-1}{2}$

State, in order, whether S_1, S_2, S_3, S_4 are true or false

(A) TFTF (B) TTTT (C) FFFF (D) FTFT

3. If $\frac{{}^nC_r+4\,{}^nC_{r+1}+6\,{}^nC_{r+2}+4\,{}^nC_{r+3}+{}^nC_{r+4}}{{}^nC_r+3\,{}^nC_{r+1}+3\,{}^nC_{r+2}+{}^nC_{r+3}}=\frac{n+k}{r+k}$ then the value of k is :

(A) 1 (B) 2 (C) 4 (D) 5

4.✍ The co-efficient of x^5 in the expansion of $(1 + x)^{21} + (1 + x)^{22} + \ldots\ldots + (1 + x)^{30}$ is :

(A) ${}^{51}C_5$ (B) ${}^{9}C_5$ (C) ${}^{31}C_6 - {}^{21}C_6$ (D) ${}^{30}C_5 + {}^{20}C_5$

5.✍ The coefficient of x^{52} in the expansion $\sum_{m=0}^{100} {}^{100}C_m\,(x-3)^{100-m}.\,2^m$ is :

(A) ${}^{100}C_{47}$ (B) ${}^{100}C_{48}$ (C) $-{}^{100}C_{52}$ (D) $-{}^{100}C_{100}$

6. The sum of the coefficients of all the integral powers of x in the expansion of $\left(1+2\sqrt{x}\right)^{40}$ is :

(A) $3^{40} + 1$ (B) $3^{40} - 1$ (C) $\frac{1}{2}(3^{40} - 1)$ (D) $\frac{1}{2}(3^{40} + 1)$

7. $\sum_{r=0}^{n}(-1)^r\,{}^nC_r.\frac{(1+r\ell n10)}{(1+\ell n10^n)^r}=$

(A) 0 (B) 1/2 (C) 1 (D) None of these

8. The coefficient of the term independent of x in the expansion of $\left(\frac{x+1}{x^{\frac{2}{3}}-x^{\frac{1}{3}}+1}-\frac{x-1}{x-x^{\frac{1}{2}}}\right)^{10}$ is :

(A) 70 (B) 112 (C) 105 (D) 210

9. Coefficient of x^{n-1} in the expansion of, $(x + 3)^n + (x + 3)^{n-1}(x + 2) + (x + 3)^{n-2}(x + 2)^2 + ... + (x + 2)^n$ is :
(A) ${}^{n+1}C_2(3)$ (B) ${}^{n-1}C_2(5)$ (C) ${}^{n+1}C_2(5)$ (D) ${}^{n}C_2(5)$

10. Let $f(n) = 10^n + 3.4^{n+2} + 5$, $n \in N$. The greatest value of the integer which divides f(n) for all n is :
(A) 27 (B) 9 (C) 3 (D) None of these

11. If $(1 + x)^n = \sum_{r=0}^{n} a_r x^r$ and $b_r = 1 + \frac{a_r}{a_{r-1}}$ and $\prod_{r=1}^{n} b_r = \frac{(101)^{100}}{100!}$, then n equals to :
(A) 99 (B) 100 (C) 101 (D) 102

12. Number of rational terms in the expansion of $\left(1+\sqrt{2}+\sqrt{5}\right)^6$ is :
(A) 7 (B) 10 (C) 6 (D) 8

13. If $S = {}^{404}C_4 - {}^4C_1 . {}^{303}C_4 + {}^4C_2 . {}^{202}C_4 - {}^4C_3 . {}^{101}C_4 = (101)^k$ then k equals to :
(A) 1 (B) 2 (C) 4 (D) 6

14. ${}^{10}C_0^{\,2} - {}^{10}C_1^{\,2} + {}^{10}C_2^{\,2} - .... - \left({}^{10}C_9\right)^2 + \left({}^{10}C_{10}\right)^2 =$
(A) 0 (B) $\left({}^{10}C_5\right)^2$ (C) $-{}^{10}C_5$ (D) $2\,{}^9C_5$

15. The sum $\sum_{r=0}^{n} (r + 1)\, C_r^2$ is equal to :
(A) $\frac{(n+2)(2n-1)!}{n!\,(n-1)!}$ (B) $\frac{(n+2)(2n+1)!}{n!\,(n-1)!}$
(C) $\frac{(n+2)(2n+1)!}{n!\,(n+1)!}$ (D) $\frac{(n+2)(2n-1)!}{n!\,(n+1)!}$

16. If $(1 + x + x^2 + x^3)^5 = a_0 + a_1x + a_2x^2 + ... + a_{15}x^{15}$, then a_{10} equals to :
(A) 99 (B) 101 (C) 100 (D) 110

17. If $a_n = \sum_{r=0}^{n} \frac{1}{{}^nC_r}$, the value of $\sum_{r=0}^{n} \frac{n-2r}{{}^nC_r}$ is :
(A) $\frac{n}{2}a_n$ (B) $\frac{1}{4}a_n$ (C) na_n (D) 0

18. The sum of: $3.{}^nC_0 - 8.{}^nC_1 + 13.{}^nC_2 - 18.{}^nC_3 + ...$ upto (n+1) terms is $(n \geq 2)$:
(A) zero (B) 1 (C) 2 (D) none of these

19. If $\sum_{r=0}^{n-1}\left(\frac{{}^nC_r}{{}^nC_r + {}^nC_{r+1}}\right)^3 = \frac{4}{5}$ then n =
(A) 4 (B) 6 (C) 8 (D) None of these

20. The number of terms in the expansion of $\left(x^2 + 1 + \frac{1}{x^2}\right)^n$, $n \in N$, is :
(A) 2n (B) 3n (C) 2n + 1 (D) 3n + 1

21. Suppose

$$\det\begin{bmatrix} \sum_{k=0}^{n} k & \sum_{k=0}^{n} {}^nC_k k^2 \\ \sum_{k=0}^{n} {}^nC_k k & \sum_{k=0}^{n} {}^nC_k 3^k \end{bmatrix} = 0$$

holds for some positive integer n. Then $\sum_{k=0}^{n} \frac{{}^nC_k}{k+1}$ equals.

PART-II: NUMERICAL VALUE QUESTIONS

INSTRUCTION :
- ❖ The answer to each question is **NUMERICAL VALUE** with two digit integer and decimal upto two digit.
- ❖ If the numerical value has more than two decimal places **truncate/round-off** the value to **TWO** decimal placed.

1. If $\frac{1}{1!10!} + \frac{1}{2!9!} + \frac{1}{3!8!} + ... + \frac{1}{10!1!} = \frac{(2^{10}-1)}{k\ 10!}$ then find the value of k.

2. If the 6th term in the expansion of $\left[\frac{1}{x^{8/3}} + x^2 \log_{10} x\right]^8$ is 5600, then x =

3. The number of values of ' x ' for which the fourth term in the expansion,

$$\left(5^{\frac{2}{5}\log_5\sqrt{4^x+44}} + \frac{1}{5^{\log_5 \sqrt[3]{2^{x-1}+7}}}\right)^8$$ is 336, is :

4. If second, third and fourth terms in the expansion of $(x + a)^n$ are 240, 720 and 1080 respectively, then ratio of last term and first term is.

5. Let the co-efficients of x^n in $(1 + x)^{2n}$ & $(1 + x)^{2n-1}$ be P & Q respectively, then $\left(\frac{P + Q}{P}\right)^4 =$

6. In the expansion of $\left(3^{\frac{-x}{4}} + 3^{\frac{5x}{4}}\right)^n$, the sum of the binomial coefficients is 256 and four times the term with greatest binomial coefficient exceeds the square of the third term by 21n, then find x.

7. If $\sum_{k=1}^{19} \frac{(-2)^k}{k!(19-k)!} = \frac{1}{k\ 18!}$ then find k.

8. The value of p, for which coefficient of x^{50} in the expression
$(1 + x)^{1000} + 2x(1 + x)^{999} + 3x^2(1 + x)^{998} + ... + 1001\, x^{1000}$ is equal to ${}^{1002}C_p$, is :

9. If { x } denotes the fractional part of ' x ', and $\left\{\frac{3^{1001}}{82}\right\} = \frac{1}{\lambda}$ then value of λ is

10. The index ' n ' of the binomial $\left(\frac{x}{5}+\frac{2}{5}\right)^n$ if the only 9th term of the expansion has numerically the greatest coefficient ($n \in N$) then find $\frac{T_9}{T_8}$ (where T_r denote coefficient of r^{th} term from beginning in the expansion)

11. Sum of square of all possible values of 'r' satisfying the equation,
$^{39}C_{3r-1} - {^{39}C_{r^2}} = {^{39}C_{r^2-1}} - {^{39}C_{3r}}$ is :

12. Find the value of
$^6C_0 . {^{12}C_6} - {^6C_1}\,{^{11}C_6} + {^6C_2}\,{^{10}C_6} - {^6C_3} . {^9C_6} + {^6C_4} . {^8C_6} - {^6C_5} . {^7C_6} + {^6C_6} . {^6C_6}$

13. If n is a positive integer & $C_k = {^nC_k}$, find the value of $\left(\sum_{k=1}^{n}\frac{k^3}{n(n+1)^2.(n+2)}\left(\frac{C_k}{C_{k-1}}\right)^2\right)$ is :

14. The value of the expression $\left(\sum_{r=0}^{10}{^{10}C_r}\right)\left(\sum_{K=0}^{10}(-1)^K\frac{^{10}C_K}{2^K}\right)$ is :

15. The value of λ if $\sum_{m=97}^{100}{^{100}C_m} . {^mC_{97}} = \lambda\ {^{99}C_{96}}$ is :

16. If $(1 + x + x^2 + ... + x^p)^n = a_0 + a_1x + a_2x^2 + ... + a_{np}\,x^{np}$, then the value of :
$\frac{1}{p(p+1)^7}\,[a_1 + 2a_2 + 3a_3 + ... + 7p\ a_{7p}]$ is :

17. If $({^{2n}C_1})^2 + 2.({^{2n}C_2})^2 + 3.({^{2n}C_3})^2 + ... + 2n.({^{2n}C_{2n}})^2 = 18 . {^{4n-1}C_{2n-1}}$, then n is :

18. If $\sum_{r=0}^{n}\frac{2r+3}{r+1}.{^nC_r} = \frac{(2n+3k)2^n-1}{n+1}$ then 'k' is

19. If $\sum_{r=0}^{n}\frac{(-1)^r.C_r}{(r+1)(r+2)(r+3)} = \frac{a}{(n+b)}$, then a + b is

20 $\sum_{k=1}^{3n}{^{6n}C_{2k-1}}(-3)^k$ is equal to :

21. If x is very large as compare to y, then the value of k in $\sqrt{\frac{x}{x+y}}\sqrt{\frac{x}{x-y}} = 1+\frac{ky^2}{x^2}$

PART - III : ONE OR MORE THAN ONE OPTIONS CORRECT TYPE

1. In the expansion of $\left(\sqrt[3]{4}+\frac{1}{\sqrt[4]{6}}\right)^{20}$

(A) the number of irrational terms is 19
(B) middle term is irrational
(C) the number of rational terms is 2
(D) 9th term is rational

2. The coefficient of x^4 in $\left(\frac{1+x}{1-x}\right)^2$, $|x| < 1$, is

(A) 4 (B) –4 (C) $10 + {}^4C_2$ (D) 16

3. $7^9 + 9^7$ is divisible by :

(A) 16 (B) 24 (C) 64 (D) 72

4. The sum of the series $\sum_{r=1}^{n}(-1)^{r-1}.\ {}^nC_r(a-r)$ is equal to :

(A) 5 if a = 5 (B) –5 if a = 5 (C) –5 if a = –5 (D) 5 if a = –5

5. Let $a_n = \frac{1000^n}{n!}$ for $n \in N$, then a_n is greatest, when

(A) n = 997 (B) n = 998 (C) n = 999 (D) n = 1000

6. ${}^nC_0 - 2.3\ {}^nC_1 + 3.3^2\ {}^nC_2 - 4.3^3\ {}^nC_3 + ... + (-1)^n\ (n+1)\ {}^nC_n\ 3^n$ is equal to

(A) $2^n\left(\frac{3n}{2}+1\right)$ if n is even (B) $2^n\left(n+\frac{3}{2}\right)$ if n is even

(C) $-\ 2^n\left(\frac{3n}{2}+1\right)$ if n is odd (D) $2^n\left(n+\frac{3}{2}\right)$ if n is odd

7. Element in set of values of r for which, ${}^{18}C_{r-2} + 2.\ {}^{18}C_{r-1} + {}^{18}C_r \geq {}^{20}C_{13}$ is :

(A) 9 (B) 5 (C) 7 (D) 10

8. The expansion of $(3x + 2)^{-1/2}$ is valid in ascending powers of x, if x lies in the interval.

(A) (0, 2/3) (B) (–3/2, 3/2) (C) (–2/3, 2/3) (D) $(-\infty, -3/2)\ (3/2, \infty)$

9. If $(1 + 2x + 3x^2)^{10} = a_0 + a_1x + a_2x^2 + ... + a_{20}x^{20}$, then :

(A) $a_1 = 20$ (B) $a_2 = 210$

(C) $a_4 = 8085$ (D) $a_{20} = 2^2.\ 3^7.\ 7$

10. In the expansion of $(x + y + z)^{25}$

(A) every term is of the form ${}^{25}C_r.\ {}^rC_k.\ x^{25-r}.\ y^{r-k}.\ z^k$ (B) the coefficient of $x^8\ y^9\ z^9$ is 0

(C) the number of terms is 325 (D) none of these

11. If $(1 + x + 2x^2)^{20} = a_0 + a_1x + a_2x^2 + ... + a_{40}x^{40}$, then $a_0 + a_2 + a_4 .... + a_{38}$ is equal to :

(A) $2^{19}\ (2^{30} + 1)$ (B) $2^{19}(2^{20} - 1)$ (C) $2^{39} - 2^{19}$ (D) $2^{39} + 2^{19}$

12. $n^n\left(\frac{n+1}{2}\right)^{2n}$ is $(n \in N)$

(A) Less than $\left(\frac{n+1}{2}\right)^3$ (B) Greater than or equal to $\left(\frac{n+1}{2}\right)^3$

(C) Less than $(n!)^3$ (D) Greater than or equal to $(n!)^3$.

13. If recursion polynomials $P_k(x)$ are defined as $P_1(x) = (x-2)^2$, $P_2(x) = ((x-2)^2 - 2)^2$
$P_3(x) = ((x-2)^2 - 2)^2 - 2)^2$ .... (In general $P_k(x) = (P_{k-1}(x) - 2)^2$, then the constant term in $P_k(x)$ is

(A) 4 (B) 2 (C) 16 (D) a perfect square

PART - IV : COMPREHENSION

Comprehension # 1 (Q. No. 1 to 3)

Consider, sum of the series $\sum\sum_{0 \le i < j \le n} f(i)\ f(j)$

In the given summation, i and j are not independent.

In the sum of series $\sum_{i=1}^{n}\sum_{j=1}^{n} f(i)\quad f(j) = \sum_{i=1}^{n}\left(f(i)\quad\left(\sum_{j=1}^{n} f(j)\right)\right)$ i and j are independent. In this summation, three types of terms occur, those when i < j, i > j and i = j.

Also, sum of terms when i < j is equal to the sum of the terms when i > j if f(i) and f(j) are symmetrical. So, in that case

$$\sum_{i=1}^{n}\sum_{j=1}^{n} f(i)f(j) = \sum\sum_{0\le i<j\le n} f(i)f(j)$$

$$+ \sum\sum_{0\le i<j\le n} f(i)f(j) + \sum\sum_{i=j} f(i)f(j)$$

$$= 2\sum\sum_{0\le i<j\le n} f(i)f(j) + \sum\sum_{i=j} f(i)f(j)$$

$$\Rightarrow \qquad \sum\sum_{0\le i<j\le n} f(i)f(j) = \frac{\sum_{i=0}^{n}\sum_{j=0}^{n} f(i)f(j) - \sum\sum_{i=j} f(i)f(j)}{2}$$

When f(i) and f(j) are not symmetrical, we find the sum by listing all the terms.

1. $\sum\sum_{0\le i<j\le n} {}^{n}C_i \quad {}^{n}C_j$ is equal to

(A) $\frac{2^{2n} - {}^{2n}C_n}{2}$ (B) $\frac{2^{2n} + {}^{2n}C_n}{2}$ (C) $\frac{2^{2n} - {}^{n}C_n}{2}$ (D) $\frac{2^{2n} + {}^{n}C_n}{2}$

2. Let ${}^{0}C_0 = 1$, then $\sum_{m=0}^{n}\ \sum_{p=0}^{m} {}^{n}C_m\ .\ {}^{m}C_p$ is equal to

(A) $2^n - 1$ (B) 3^n (C) $3^n - 1$ (D) 2^n

3. $\sum\sum_{0\le i\le j\le n}\left({}^{n}C_i + {}^{n}C_j\right)$

(A) $(n + 2)2^n$ (B) $(n + 1)2^n$ (C) $(n - 1)2^n$ (D) $(n + 1)2^n - 1$

Comprehension # 2 (Q. No. 4 to 6)

Let P be a product given by $P = (x + a_1)(x + a_2) \ldots\ldots\ldots (x + a_n)$

and Let $S_1 = a_1 + a_2 + \ldots\ldots. + a_n = \sum_{i=1}^{n} a_i$, $S_2 = \sum\sum_{i<j} a_i.a_j$, $S_3 = \sum\sum_{i<j<k}\sum a_i.a_j.a_k$ and so on,

then it can be shown that

$P = x^n + S_1 x^{n-1} + S_2 x^{n-2} + \ldots\ldots\ldots + S_n$.

4. The coefficient of x^8 in the expression $(2 + x)^2 (3 + x)^3 (4 + x)^4$ must be

(A) 26 (B) 27 (C) 28 (D) 29

5. The coefficient of x^{203} in the expression $(x - 1)(x^2 - 2)(x^3 - 3)$ .... $(x^{20} - 20)$ must be
(A) 11 (B) 12 (C) 13 (D) 15

6. The coefficient of x^{98} in the expression of $(x - 1)(x - 2)$ .... $(x - 100)$ must be
(A) $1^2 + 2^2 + 3^2 + \ldots\ldots + 100^2$
(B) $(1 + 2 + 3 + \ldots\ldots + 100)^2 - (1^2 + 2^2 + 3^2 + \ldots\ldots + 100^2)$
(C) $\frac{1}{2}[(1 + 2 + 3 + \ldots\ldots + 100)^2 - (1^2 + 2^2 + 3^2 + \ldots\ldots + 100^2)]$
(D) None of these

Comprehension # 3 (Q.No. 7 to 9)

Let $(7 + 4\sqrt{3})^n = I + f = {}^nC_0.7^n + {}^nC_1.7^{n-1}.(4\sqrt{3})^1 + \ldots\ldots$ ....(i)
where I & f are its integral and fractional parts respectively.
It means $0 < f < 1$
Now, $0 < 7 - 4\sqrt{3} < 1 \Rightarrow 0 < (7 - 4\sqrt{3})^n < 1$
Let $(7 - 4\sqrt{3})^n = f' = {}^nC_0.7^n - {}^nC_1.7^{n-1}.(4\sqrt{3})^1 + \ldots\ldots$ ....(ii)
$\Rightarrow$ $0 < f' < 1$
Adding (i) and (ii) (so that irrational terms cancelled out)
$I + f + f' = (7 + 4\sqrt{3})^n + (7 - 4\sqrt{3})^n$
$= 2[{}^nC_0\, 7^n + {}^nC_2\, 7^{n-2}(4\sqrt{3})^2 + \ldots\ldots\ldots]$
$I + f + f'$ = even integer $\Rightarrow$ ($f + f'$ must be an integer)
$0 < f + f' < 2 \Rightarrow f + f' = 1$
with help of above analysis answer the following questions

7. If $\left(3\sqrt{3} + 5\right)^n = p + f$, where p is an integer and f is a proper fraction, then find the value of $\left(3\sqrt{3} - 5\right)^n$, $n \in N$, is
(A) 1 – f, if n is even (B) f, if n is even (C) 1 – f, if n is odd (D) f, if n is odd

8. If $\left(9 + \sqrt{80}\right)^n = I + f$, where I, n are integers and $0 < f < 1$, then :
(A) I is an odd integer (B) I is an even integer
(C) $(I + f)(1 - f) = 1$ (D) $1 - f = \left(9 - \sqrt{80}\right)^n$

9. The integer just above $(\sqrt{3} + 1)^{2n}$ is, for all $n \in N$.
(A) divisible by 2^n (B) divisible by 2^{n+1} (C) divisible by 8 (D) divisible by 16

Exercise-3

✍ Marked questions are recommended for Revision.
*** Marked Questions may have more than one correct option.**

PART - I : JEE (ADVANCED) / IIT-JEE PROBLEMS (PREVIOUS YEARS)

1. Coefficient of t^{24} in $(1 + t^2)^{12}(1 + t^{12})(1 + t^{24})$ is: **[IIT-JEE-2003, Scr, (3, – 1), 84]**
(A) ${}^{12}C_6 + 3$ (B) ${}^{12}C_6 + 1$ (C) ${}^{12}C_6$ (D) ${}^{12}C_6 + 2$

2.✍ Prove that $2^k \binom{n}{0}\binom{n}{k} - 2^{k-1}\binom{n}{1}\binom{n-1}{k-1} + 2^{k-2}\binom{n}{2}\binom{n-2}{k-2} - \ldots\ldots + (-1)^k\binom{n}{k}\binom{n-k}{0} = \binom{n}{k}$.
[IIT-JEE-2003, Main, (2, 0), 60]

3. If ${}^{(n-1)}C_r = (k^2 - 3)\ {}^nC_{r+1}$, then an interval in which k lies is **[IIT-JEE-2004, Scr, (3, – 1), 84]**

(A) $(2, \infty)$ (B) $(-\infty, -2)$ (C) $\left[-\sqrt{3}, \sqrt{3}\right]$ (D) $\left(\sqrt{3}, 2\right]$

4. The value of **[IIT-JEE-2005, Scr, (3, – 1), 84]**

$$\binom{30}{0}\binom{30}{10} - \binom{30}{1}\binom{30}{11} + \binom{30}{2}\binom{30}{12} - \ldots\ldots\ldots + \binom{30}{20}\binom{30}{30} \text{ is :}$$

(A) $\binom{60}{20}$ (B) $\binom{30}{10}$ (C) $\binom{30}{15}$ (D) None of these

5. For r = 0, 1, ..., 10, let A_r, B_r and C_r denote, respectively, the coefficient of x^r in the expansions of $(1 + x)^{10}$, $(1 + x)^{20}$ and $(1 + x)^{30}$. Then $\sum_{r=1}^{10} A_r(B_{10}B_r - C_{10}A_r)$ is equal to

[IIT-JEE 2010, Paper-2, (5, –2)/79]

(A) $B_{10} - C_{10}$ (B) $A_{10}(B^2_{10} - C_{10}A_{10})$ (C) 0 (D) $C_{10} - B_{10}$

6. The coefficients of three consecutive terms of $(1 + x)^{n+5}$ are in the ratio 5 : 10 : 14. Then n =
[JEE (Advanced) 2013, Paper-1, (4, – 1)/60]

7. Coefficient of x^{11} in the expansion of $(1 + x^2)^4 (1 + x^3)^7 (1 + x^4)^{12}$ is
[JEE (Advanced) 2014, Paper-2, (3, –1)/60]

(A) 1051 (B) 1106 (C) 1113 (D) 1120

8. The coefficient of x^9 in the expansion of $(1 + x)(1 + x^2)(1 + x^3)\ldots\ldots(1 + x^{100})$ is
[JEE (Advanced) 2015, P-2 (4, 0) / 80]

9. Let m be the smallest positive integer such that the coefficient of x^2 in the expansion of $(1 + x)^2 + (1 + x)^3 + \ldots\ldots\ldots + (1 + x)^{49} + (1 + mx)^{50}$ is $(3n + 1)\ {}^{51}C_3$ for some positive integer n. Then the value of n is **[JEE (Advanced) 2016, Paper-1, (3, 0)/62]**

10. Let $X = ({}^{10}C_1)^2 + 2({}^{10}C_2)^2 + 3({}^{10}C_3)^2 + \ldots\ldots + 10({}^{10}C_{10})^2$ where ${}^{10}C_r$, $r \in \{1, 2, \ldots\ldots, 10\}$ denote binomial coefficients. Then the value of $\frac{1}{1430}$ X is _______ . **[JEE (Advanced) 2018, Paper-1, (3, 0)/60]**

PART - II : JEE (MAIN) / AIEEE PROBLEMS (PREVIOUS YEARS)

1. Let $S_1 = \sum_{j=1}^{10} j\ (j-1)\ {}^{10}C_j$, $S_2 = \sum_{j=1}^{10} j\ {}^{10}C_j$ and $S_3 = \sum_{j=1}^{10} j^2\ {}^{10}C_j$. **[AIEEE 2009, (4, –1), 144]**

Statement -1 : $S_3 = 55 \times 2^9$.

Statement -2 : $S_1 = 90 \times 2^8$ and $S_2 = 10 \times 2^8$.

(1) Statement -1 is true, Statement-2 is true ; Statement -2 is not a correct explanation for Statement 1.
(2) Statement-1 is true, Statement-2 is false.
(3) Statement -1 is false, Statement -2 is true.
(4) Statement -1 is true, Statement -2 is true; Statement-2 is a correct explanation for Statement-1.

2. The coefficient of x^7 in the expansion of $(1 - x - x^2 + x^3)^6$ is : **[AIEEE 2011, (4, –1), 120]**

(1) 144 (2) – 132 (3) – 144 (4) 132

3. If n is a positive integer, then $\left(\sqrt{3}+1\right)^{2n} - \left(\sqrt{3}-1\right)^{2n}$ is : **[AIEEE 2012, (4, –1), 120]**

(1) an irrational number (2) an odd positive integer
(3) an even positive integer (4) a rational number other than positive integers

4. The term independent of x in expansion of $\left(\frac{x+1}{x^{2/3}-x^{1/3}+1}-\frac{x-1}{x-x^{1/2}}\right)^{10}$ is :
[AIEEE - 2013, (4, –1),120]
(1) 4 (2) 120 (3) 210 (4) 310

5. If the coefficients of x^3 and x^4 in the expansion of $(1 + ax + bx^2)(1 - 2x)^{18}$ in powers of x are both zero, then (a, b) is equal to **[JEE(Main) 2014, (4, – 1), 120]**
(1) $\left(14,\frac{272}{3}\right)$ (2) $\left(16,\frac{272}{3}\right)$ (3) $\left(16,\frac{251}{3}\right)$ (4) $\left(14,\frac{251}{3}\right)$

6. The sum of coefficients of integral powers of x in the binomial expansion of $(1 - 2\sqrt{x})^{50}$ is
[JEE(Main) 2015, (4, – 1), 120]
(1) $\frac{1}{2}(3^{50}+1)$ (2) $\frac{1}{2}(3^{50})$ (3) $\frac{1}{2}(3^{50}-1)$ (4) $\frac{1}{2}(2^{50}+1)$

7. If the number of terms in the expansion of $\left(1-\frac{2}{x}+\frac{4}{x^2}\right)^n$, $x \neq 0$, is 28, then the sum of the coefficients of all the terms in this expansion, is **[JEE(Main) 2016, (4, – 1), 120]**
(1) 2187 (2) 243 (3) 729 (4) 64

8. The value of $({}^{21}C_1 - {}^{10}C_1) + ({}^{21}C_2 - {}^{10}C_2) + ({}^{21}C_3 - {}^{10}C_3) + ({}^{21}C_4 - {}^{10}C_4) + \ldots\ldots.. + ({}^{21}C_{10} - {}^{10}C_{10})$ is
[JEE(Main) 2017, (4, – 1), 120]
(1) $2^{21} - 2^{11}$ (2) $2^{21} - 2^{10}$ (3) $2^{20} - 2^9$ (4) $2^{20} - 2^{10}$

9. The sum of the co-efficients of all odd degree terms in the expansion of $\left(x+\sqrt{x^3-1}\right)^5+\left(x-\sqrt{x^3-1}\right)^5$, $(x > 1)$ is : **[JEE(Main) 2018, (4, – 1), 120]**
(1) 1 (2) 2 (3) –1 (4) 0

10. If the fractional part of the number $\frac{2^{403}}{15}$ is $\frac{k}{15}$, then k is equal to :
[JEE(Main) 2019, Online (09-01-19),P-1 (4, – 1), 120]
(1) 14 (2) 8 (3) 6 (4) 4

11. If $\sum_{i=1}^{20}\left(\frac{{}^{20}C_{i-1}}{{}^{20}C_i+{}^{20}C_{i-1}}\right)^3=\frac{k}{21}$, then k equals : **[JEE(Main) 2019, Online (10-01-19),P-1 (4, – 1), 120]**
(1) 50 (2) 400 (3) 200 (4) 100

12. If $\sum_{r=0}^{25}\left\{{}^{50}C_r.{}^{50-r}C_{25-r}\right\}=K\left({}^{50}C_{25}\right)$, then K is equal to :
[JEE(Main) 2019, Online (10-01-19),P-2 (4, – 1), 120]
(1) 2^{25} (2) $2^{25} - 1$ (3) $(25)^2$ (4) 2^{24}

13. Let $S_n = 1 + q + q^2 + \ldots. + q^n$ and $T_n = 1 + \left(\frac{q+1}{2}\right) + \left(\frac{q+1}{2}\right)^2 + \ldots. + \left(\frac{q+1}{2}\right)^n$.
where q is a real number and $q \neq 1$. If ${}^{101}C_1 + {}^{101}C_2 . S_1 + \ldots..+ {}^{101}C_{101} . S_{100} = \alpha T_{100}$ then α is equal to
[JEE(Main) 2019, Online (11-01-19),P-2 (4, – 1), 120]
(1) 200 (2) 2^{99} (3) 2^{100} (4) 202

Answers

EXERCISE - 1

PART -I

Section (A) :

A-1. (i) $\left(\frac{2}{x}\right)^5 - 5\left(\frac{2}{x}\right)^3 + 10\left(\frac{2}{x}\right) - 10\left(\frac{x}{2}\right) + 5\left(\frac{x}{2}\right)^3 - \left(\frac{x}{2}\right)^5$

(ii) $y^8 + 8y^5 + 24y^2 + \frac{32}{y} + \frac{16}{y^4}$

A-2. n = 9 **A-3.** 7 **A-4.** (i) $^9C_3$ (ii) $-2^7 . {}^{12}C_7$

A-5. $^{11}C_5 \frac{a^6}{b^5}, {}^{11}C_6 \frac{a^5}{b^6}$, a b = 1 **A-6.** $\frac{17}{54}$. **A-7.** (i) 171 (ii) –438

A-8. 15

Section (B) :

B-1. (i) $-\frac{35x}{y}, \frac{35y}{x}$ (ii) $(-1)^n \frac{(2n)!}{n!\ \ n!} x^n$

B-3. (i) 4 (iii) 3, 03, 803

B-4. 101^{50}

B-5. (i) $T_4 = -455 \times 3^{12}$ and $T_5 = 455 \times 3^{12}$ (ii) 22

B-6. (i) T_4 (ii) T_5, T_6 (iii) T_5 (iv) T_6

Section (D) :

D-1. $\frac{15015}{16}$ **D-2.** (i) 142 (ii) – 197 **D-4.** (i) 280 (ii) 2^5

D-5. 20

PART -II

Section (A) :

A-1. (C) **A-2.** (C) **A-3.** (A) **A-4.** (B) **A-5.** (A) **A-6.** (A) **A-7.** (B)
A -8. (C) **A-9.** (C) **A-10.** (B)

Section (B) :

B-1. (B) **B-2.** (C) **B-3.** (D) **B-4.** (A) **B-5.** (A) **B-6.** (A) **B-7.** (A)
B-8. (B)

Section (C) :

C-1. (B) **C-2.** (C) **C-3.** (C) **C-4.** (B)

Section (D) :

D-1. (D) **D-2.** (D) **D-3.** (A)

PART -III

1. (A) $\rightarrow$ (q, s), (B) $\rightarrow$ (q,s), (C) $\rightarrow$ (s), (D) $\rightarrow$ (p, s)

EXERCISE - 2

PART -I

1.	(B)	**2.**	(A)	**3.**	(C)	**4.**	(C)	**5.**	(B)	**6.**	(D)	**7.**	(A)
8.	(D)	**9.**	(C)	**10.**	(B)	**11.**	(B)	**12.**	(B)	**13.**	(C)	**14.**	(C)
15.	(A)	**16.**	(B)	**17.**	(D)	**18.**	(A)	**19.**	(A)	**20.**	(C)		
21.	(6.20)												

PART -II

1.	05.50	**2.**	10.00	**3.**	02.00	**4.**	07.59	**5.**	05.06	**6.**	00.50	**7.**	09.50
8.	50.00	**9.**	27.33	**10.**	01.25	**11.**	34.00	**12.**	01.00	**13.**	00.08	**14.**	01.00
15.	08.24 or 08.25			**16.**	03.50	**17.**	09.00	**18.**	01.33	**19.**	03.50	**20.**	00.00
21.	00.50												

PART -III

1.	(ABCD)	**2.**	(CD	**3.**	(AC)	**4.**	(AC)	**5.**	(CD)	**6.**	(AC)	**7.**	(ACD)
8.	(AC)	**9.**	(ABC)	**10.**	(AB)	**11.**	(BC)	**12.**	(BD)	**13.**	(AD)		

PART -IV

1.	(A)	**2.**	(B)	**3.**	(A)	**4.**	(D)	**5.**	(C)	**6.**	(C)	**7.**	(AD)
8.	(ACD)	**9.**	(ABC)										

EXERCISE - 3

PART -I

1.	(D)	**3.**	(D)	**4.**	(B)	**5.**	(D)	**6.**	6	**7.**	(C)	**8.**	8
9.	5	**10.**	(646)										

PART -II

1.	(2)	**2.**	(3)	**3.**	(1)	**4.**	(3)	**5.**	(2)	**6.**	(1)	**7.**	(3)
8.	(4)	**9.**	(2)	**10.**	(2)	**11.**	(4)	**12.**	(1)	**13.**	(3)		

High Level Problems (HLP)

SUBJECTIVE QUESTIONS

1. Find the coefficient of x^{49} in

$\left(x+\frac{C_1}{C_0}\right)\left(x+2^2\frac{C_2}{C_1}\right)\left(x+3^2\frac{C_3}{C_2}\right)........\left(x+50^2\frac{C_{50}}{C_{49}}\right)$ where $C_r = {}^{50}C_r$

2. The expression, $\left(\sqrt{2x^2+1}+\sqrt{2x^2-1}\right)^6+\left(\frac{2}{\sqrt{2x^2+1}+\sqrt{2x^2-1}}\right)^6$ is a polynomial of degree

3. Find the co-efficient of x^5 in the expansion of $(1 + x^2)^5 (1 + x)^4$.

4. Prove that the co-efficient of x^{15} in $(1 + x + x^3 + x^4)^n$ is $\sum_{r=0}^{5} {}^nC_{15-3r}\,{}^nC_r$.

5. If n is even natural and coefficient of x^r in the expansion of $\frac{(1+x)^n}{1-x}$ is 2^n, $(|x| < 1)$, then prove that $r \ge n$

6. Find the coefficient of x^n in polynomial $(x + {}^{2n+1}C_0)(x + {}^{2n+1}C_1)........(x + {}^{2n+1}C_n)$.

7. Find the value of $\sum_{r=1}^{n}\left(\sum_{p=0}^{r-1} {}^nC_r\,{}^rC_p\,2^p\right)$.

Comprehension (Q-8 to Q.10)

For k, n $\in$ N, we define

B(k, n) = 1.2.3......... k + 2.3.4........(k+1) + ...+ n(n + 1).......(n + k – 1), $S_0(n) = n$ and $S_k(n) = 1^k + 2^k + ... + n^k$.

To obtain value B(k, n), we rewrite B(k, n) as follows

$$B(k,n) = k!\left[{}^kC_k + {}^{k+1}C_k + {}^{k+2}C_k + ... + {}^{n+k-1}C_k\right] = k!\left({}^{n+k}C_{k+1}\right) = \frac{n(n+1)........(n+k)}{k+1}$$

where $\;{}^nC_k = \frac{n!}{k!\,(n-k)!}$

8. Prove that $S_2(n) + S_1(n) = B(2, n)$

9. Prove that $S_3(n) + 3S_2(n) = B(3, n) – 2B(1, n)$

10. If $(1 + x)^p = 1 + {}^pC_1 x + {}^pC_2 x^2 + ... + {}^pC_p x^p$, $p \in N$, then show that ${}^{k+1}C_1 S_k(n) + {}^{k+1}C_2 S_{k-1}(n) + ... + {}^{k+1}C_k S_1(n) + {}^{k+1}C_{k+1} S_0(n) = (n + 1)^{k+1} – 1$

11. Show that $25^n – 20^n – 8^n + 3^n$, $n \in I^+$ is divisible by 85.

12. Prove that ${}^nC_1\,({}^nC_2)^2\,({}^nC_3)^3........({}^nC_n)^n \le \left(\dfrac{2^n}{n+1}\right)^{{}^{n+1}C_2}$.

13. If p is nearly equal to q and n > 1, show that $\dfrac{(n+1)\;\;p+(n-1)q}{(n-1)p+(n+1)q} = \left(\dfrac{p}{q}\right)^{1/n}$. Hence find the approximate value of $\left(\dfrac{99}{101}\right)^{1/6}$.

14. If $(18x^2 + 12x + 4)^n = a_0 + a_1x + a_2x^2 + ... + a_{2n}\,x^{2n}$, then prove that
$a_r = 2^n\,3^r\,\left({}^{2n}C_r + {}^nC_1\;\;{}^{2n-2}C_r\;\; + \;{}^nC_2\;\;{}^{2n-4}C_r\;\; + \;\;...\right)$

15. Prove that $1^2.\,C_0 + 2^2.\,C_1 + 3^2.\,C_2 + 4^2.\,C_3 +.... + (n+1)^2\,C_n = 2^{n-2}\,(n+1)\,(n+4)$.

16. If $(1 - x)^{-n} = a_0 + a_1\,x + a_2\,x^2 + a_3\,x^3 +........$, find the value of, $a_0 + a_1 + a_2 +....... + a_n$.

17. Find the remainder when $32^{32^{32}}$ is divided by 7.

18. If n is an integer greater than 1, show that : $a - {}^nC_1(a-1) + {}^nC_2(a-2) -..... + (-1)^n\,(a - n) = 0$.

19. If $(1 + x)^n = p_0 + p_1\,x + p_2\,x^2 + p_3\,x^3 +.......$, then prove that :

(a) $p_0 - p_2 + p_4 -....... = 2^{n/2}\cos\dfrac{n\,\pi}{4}$ (b) $p_1 - p_3 + p_5 -....... = 2^{n/2}\sin\dfrac{n\,\pi}{4}$

20. Show that if the greatest term in the expansion of $(1 + x)^{2n}$ has also the greatest co-efficient, then ' x ' lies between, $\dfrac{n}{n+1}$ & $\dfrac{n+1}{n}$.

21. Prove that if ' p ' is a prime number greater than 2, then $\left[(2+\sqrt{5})^p\right] - 2^{p+1}$ is divisible by p, where [.] denotes greatest integer function.

22. If $\sum\limits_{r=0}^{n}\;(-1)^r.\,{}^nC_r\left[\dfrac{1}{2^r}+\dfrac{3^r}{2^{2r}}+\dfrac{7^r}{2^{3r}}+........\;\;\text{to m terms}\;\right] = k\left(1-\dfrac{1}{2^{mn}}\right)$, then find the value of k.

23. Given $s_n = 1 + q + q^2 +..... + q^n$ & $S_n = 1 + \dfrac{q+1}{2} + \left(\dfrac{q+1}{2}\right)^2 +.... + \left(\dfrac{q+1}{2}\right)^n$, $q \ne 1$,
prove that ${}^{n+1}C_1 + {}^{n+1}C_2.s_1 + {}^{n+1}C_3.s_2 +....+ {}^{n+1}C_{n+1}.s_n = 2^n.\,S_n$.

24. If $(1+x)^{15} = C_0 + C_1.\,x + C_2.\,x^2 +.... + C_{15}.\,x^{15}$, then find the value of : $C_2 + 2C_3 + 3C_4 +.... + 14C_{15}$

25. Prove that, $\dfrac{1}{2}\,{}^nC_1 - \dfrac{2}{3}\,{}^nC_2 + \dfrac{3}{4}\,{}^nC_3 - \dfrac{4}{5}\,{}^nC_4 +..... + \dfrac{(-1)^{n+1}\,n}{n+1}.\,{}^nC_n = \dfrac{1}{n+1}$

26. Prove that $\sum\limits_{r=0}^{n} r^2\;\;{}^nC_r\,p^r\,q^{n-r} = npq + n^2p^2$, if p + q = 1.

27. Prove that : $(n-1)^2 . C_1 + (n-3)^2 . C_3 + (n-5)^2 . C_5 + ... = n(n+1)2^{n-3}$

28. Prove that ${}^{n}C_r + 2\ {}^{n+1}C_r + 3\ {}^{n+2}C_r + ... + (n+1)\ {}^{2n}C_r = {}^{n}C_{r+2} + (n+1)\ {}^{2n+1}C_{r+1} - {}^{2n+1}C_{r+2}$

29. Show that, $\sqrt{3} = 1 + \frac{1}{3} + \frac{1}{3}.\frac{3}{6} + \frac{1}{3}.\frac{3}{6}.\frac{5}{9} + \frac{1}{3}.\frac{3}{6}.\frac{5}{9}.\frac{7}{12} + ...$

30. If $(1+x)^n = C_0 + C_1x + C_2x^2 + ... + C_nx^n$, show that for $m \geq 2$
$C_0 - C_1 + C_2 - .... + (-1)^{m-1}C_{m-1} = (-1)^{m-1}\ {}^{n-1}C_{m-1}$.

31. If $(1+x)^n = C_0 + C_1x + C_2x^2 + ... + C_n x^n$, then show that the sum of the products of the C_i's taken two at a time, represented by $\underset{0 \leq i < j \leq n}{\sum\sum} C_i C_j$ is equal to $2^{2n-1} - \frac{2n\,!}{2\,(n\,!)^2}$.

32. If $a_0, a_1, a_2,.....$ be the coefficients in the expansion of $(1 + x + x^2)^n$ in ascending powers of x, then prove that :

(i) $a_0 a_1 - a_1 a_2 + a_2 a_3 - .... = 0$

(ii) $a_0a_2 - a_1a_3 + a_2a_4 - ..... + a_{2n-2}\ a_{2n} = a_{n+1}$

(iii) $E_1 = E_2 = E_3 = 3^{n-1}$; where $E_1 = a_0 + a_3 + a_6 + ...$; $E_2 = a_1 + a_4 + a_7 + ...$ & $E_3 = a_2 + a_5 + a_8 + ...$

Answers

1.	22100	**2.**	6	**3.**	60	**6.**	2^{2n}	**7.**	$4^n - 3^n$
13.	$\frac{1198}{1202}$	**16.**	$\frac{(2n)!}{(n!)^2}$	**17.**	4	**22.**	$\frac{1}{2^n - 1}$	**24.**	212993

Exercise-1

✍ **Marked questions are recommended for Revision.**
✍ **चिन्हित प्रश्न दोहराने योग्य प्रश्न है।**

PART - I : SUBJECTIVE QUESTIONS

भाग - I : विषयात्मक प्रश्न (SUBJECTIVE QUESTIONS)

Section (A) : General Term & Coefficient of x^k in $(ax+b)^n$
खण्ड (A) : व्यापक पद एवं $(ax+b)^n$ में x^k का गुणांक

A-1. Expand the following :
निम्न का प्रसार करो :

(i) $\left(\frac{2}{x}-\frac{x}{2}\right)^5$, $(x \neq 0)$ (ii) $\left(y^2+\frac{2}{y}\right)^4$, $(y \neq 0)$

Ans. (i) $\left(\frac{2}{x}\right)^5 - 5\left(\frac{2}{x}\right)^3 + 10\left(\frac{2}{x}\right) - 10\left(\frac{x}{2}\right) + 5\left(\frac{x}{2}\right)^3 - \left(\frac{x}{2}\right)^5$

(ii) $y^8 + 8y^5 + 24y^2 + \frac{32}{y} + \frac{16}{y^4}$

Sol. **(i)** $\left(\frac{2^5}{x}\right) - {}^5C_1\left(\frac{2}{x}\right)^4\left(\frac{x}{2}\right)^1 + {}^5C_2\left(\frac{2}{x}\right)^3\left(\frac{x}{2}\right)^2 - {}^5C_3\left(\frac{2}{x}\right)^2\left(\frac{x}{2}\right)^3 + {}^5C_4\left(\frac{2}{x}\right)^1\left(\frac{x}{2}\right)^4 - {}^5C_5\left(\frac{x}{2}\right)^5$

$= \left(\frac{2}{x}\right)^5 - 5\left(\frac{2}{x}\right)^3 + 10\left(\frac{2}{x}\right) - 10\left(\frac{x}{2}\right) + 5\left(\frac{x}{2}\right)^3 - \left(\frac{x}{2}\right)^5$

(ii) $(y^2)^4 + {}^4C_1(y^2)^3(2/y) + {}^4C_2(y^2)^2(2/y)^2 + {}^4C_3(y^2)(2/y)^3 + {}^4C_4(2/y)^4 = y^8 + 8y^5 + 24y^2 + \frac{32}{y} + \frac{16}{y^4}$

A-2. In the binomial expansion of $\left(\sqrt[3]{2}+\frac{1}{\sqrt[3]{3}}\right)^n$, the ratio of the 7th term from the begining to the 7th term from the end is 1 : 6 ; find n.

$\left(\sqrt[3]{2}+\frac{1}{\sqrt[3]{3}}\right)^n$ के प्रसार में प्रारम्भ से 7 वें पद और अंत से 7 वें पद का अनुपात 1 : 6 है, तब n का मान ज्ञात करो।

Ans. n = 9

Sol. 7th term from beginning $T_7 = {}^nC_6(2)^{\frac{n-6}{3}}\left(\frac{1}{3}\right)^2$

7th term from the end $T_{n-5} = {}^nC_{n-6}(2)^2\left(\frac{1}{3}\right)^{\frac{n-6}{3}}$

$$\frac{T_7}{T_{n-5}} = \frac{1}{6} = \frac{2^{\left(\frac{n-6}{3}-2\right)}}{\left(\frac{1}{3}\right)^{\left(\frac{n-6}{3}-2\right)}} \Rightarrow \frac{1}{6} = (6)^{\frac{n-12}{3}} \Rightarrow \frac{n-12}{3} = -1 \Rightarrow n = 9$$

Hindi प्रारम्भ से 7वाँ पद $T_7 = {}^nC_6(2)^{\frac{n-6}{3}}\left(\frac{1}{3}\right)^2$

अन्त से 7वाँ पद $T_{n-5} = {}^nC_{n-6}(2)^2\left(\frac{1}{3}\right)^{\frac{n-6}{3}}$

$$\frac{T_7}{T_{n-5}} = \frac{1}{6} = \frac{2^{\left(\frac{n-6}{3}-2\right)}}{\left(\frac{1}{3}\right)^{\left(\frac{n-6}{3}-2\right)}} \Rightarrow \frac{1}{6} = (6)^{\frac{n-12}{3}} \Rightarrow \frac{n-12}{3} = -1 \Rightarrow n = 9$$

A-3. Find the degree of the polynomial $\left(x+(x^3-1)^{1/2}\right)^5 + \left(x-(x^3-1)^{1/2}\right)^5$.

बहुपद $\left(x+(x^3-1)^{1/2}\right)^5 + \left(x-(x^3-1)^{1/2}\right)^5$. की घात ज्ञात कीजिए।

Ans. 7

Sol. $= 2[x^5 + {}^5C_2 \cdot x^3(x^3-1)^{1/2})^2 + {}^5C_4 \cdot x\,((x^3-1)^{1/2})^4]$

$= 2[x^5 + 10x^3(x^3-1) + {}^5C_4 . x\,(x^3-1)^2]$

$= 2[x^5 + 10x^6 - 10x^3 + 5x\,(x^6 - 2x^3 + 1)] \Rightarrow$ degree is 7 (घात 7 है)

A-4. Find the coefficient of

(i) x^6y^3 in $(x + y)^9$ (ii) $a^5 b^7$ in $(a - 2b)^{12}$

गुणांक का मान ज्ञात करो–

(i) $(x + y)^9$ में x^6y^3 का (ii) $(a - 2b)^{12}$ में $a^5 b^7$ का

Ans. (i) ${}^9C_3$ (ii) $-2^7 . {}^{12}C_7$

Sol. **(i)** $(x + y)^9 = \sum_{r=0}^{9} {}^9C_r x^{9-r} y^r$ $\therefore$ co-efficient of $x^6 y^3 = {}^9C_3$

(ii) $(a - 2b)^{12} = \sum_{r=0}^{12} {}^{12}C_r a^{12-r}(-2b)^r$ $\therefore$ Co-efficient of $a^5 b^7 = {}^{12}C_7(-2)^7$

Hindi. **(i)** $(x + y)^9 = \sum_{r=0}^{9} {}^9C_r x^{9-r} y^r$ $\therefore$ $x^6 y^3$ का गुणांक $= {}^9C_3$

(ii) $(a - 2b)^{12} = \sum_{r=0}^{12} {}^{12}C_r a^{12-r}(-2b)^r$ $\therefore$ $a^5 b^7$ का गुणांक $= {}^{12}C_7(-2)^7$

A-5. Find the co-efficient of x^7 in $\left(ax^2 + \frac{1}{b\ x}\right)^{11}$ and of x^{-7} in $\left(ax - \frac{1}{b\ x^2}\right)^{11}$ and find the relation between 'a' & 'b' so that these co-efficients are equal. (where $a, b \neq 0$).

$\left(ax^2 + \frac{1}{b\ x}\right)^{11}$ के प्रसार में x^7 का गुणांक और $\left(ax - \frac{1}{b\ x^2}\right)^{11}$ के प्रसार में x^{-7} का गुणांक ज्ञात करो। यदि ये गुणांक परस्पर बराबर हो, तो 'a' एवं 'b' के बीच सम्बन्ध ज्ञात करो(जहाँ $a, b \neq 0$)

Ans. ${}^{11}C_5 \frac{a^6}{b^5}$, ${}^{11}C_6 \frac{a^5}{b^6}$, $a\,b = 1$

Sol. Co-efficient of x^7 in $\left(ax^2 + \frac{1}{bx}\right)^{11}$

General Term $= {}^{11}C_r (ax^2)^{11-r} \left(\frac{1}{bx}\right)^r = {}^{11}C_r a^{11-r} b^{-r} x^{22-3r}$

Put $22 - 3r = 7$

$r = 5 \quad \therefore \quad$ Co-efficient of $x^7 = {}^{11}C_5\, a^6 . b^{-5}$

Co-efficient of x^{-7} in $\left(ax - \frac{1}{bx^2}\right)^{11}$

General Term $= {}^{11}C_r\ (ax)^{11-r} \left(-\frac{1}{bx^2}\right)^r = {}^{11}C_r\, a^{11-r} (b)^{-r} (-1)^r x^{11-3r}$

Put $11 - 3r = -7 \Rightarrow r = 6 \quad \therefore \quad$ Co-efficient of $x^{-7} = {}^{11}C_6\, a^5 b^{-6}$

Given that ${}^{11}C_5\, a^6 b^{-5} = {}^{11}C_6\, a^5 b^{-6} \Rightarrow ab = 1$

Hindi. $\left(ax^2 + \frac{1}{bx}\right)^{11}$ के प्रसार में x^7 का गुणांक

व्यापक पद $= {}^{11}C_r\, (ax^2)^{11-r} \left(\frac{1}{bx}\right)^r = {}^{11}C_r\, a^{11-r} b^{-r} x^{22-3r}$

$22 - 3r = 7$ रखने पर

$r = 5 \quad \therefore \quad x^7$ का गुणांक $= {}^{11}C_5\, a^6 . b^{-5}$

$\left(ax - \frac{1}{bx^2}\right)^{11}$ के प्रसार में x^{-7} का गुणांक

व्यापक पद $= {}^{11}C_r\ (ax)^{11-r} \left(-\frac{1}{bx^2}\right)^r = {}^{11}C_r\, a^{11-r} (b)^{-r} (-1)^r x^{11-3r}$

$11 - 3r = -7$ रखने पर $\Rightarrow r = 6 \quad \therefore \quad x^{-7}$ का गुणांक $= {}^{11}C_6\, a^5 b^{-6}$

दिया गया है $\quad {}^{11}C_5\, a^6 b^{-5} = {}^{11}C_6\, a^5 b^{-6} \Rightarrow ab = 1$

A-6. Find the term independent of 'x' in the expansion of the expression,

$(1 + x + 2x^3) \left(\frac{3}{2}x^2 - \frac{1}{3\ x}\right)^9$.

$(1 + x + 2x^3) \left(\frac{3}{2}x^2 - \frac{1}{3\ x}\right)^9$ के प्रसार में x से स्वतंत्र पद ज्ञात कीजिए।

Ans $\frac{17}{54}$.

Sol. Co-efficient of x^0 in $(1 + x + 2x^3) \left(\frac{3}{2}x^2 - \frac{1}{3x}\right)^9$

$=$ Co-efficient of x^0 in $\left(\frac{3}{2}x^2 - \frac{1}{3x}\right)^9$ + Co-efficient of x^{-1} in $\left(\frac{3}{2}x^2 - \frac{1}{3x}\right)^9$

+ 2 Co-efficient of x^{-3} in $\left(\frac{3}{2}x^2 - \frac{1}{3x}\right)^9$

$$= {}^9C_{r_1}\left(\frac{3}{2}x^2\right)^{9-r_1}\left(-\frac{1}{3x}\right)^{r_1} + {}^9C_{r_2}\left(\frac{3}{2}x^2\right)^{9-r_2}\left(-\frac{1}{3x}\right)^{r_2} + 2\ {}^9C_{r_3}\left(\frac{3}{2}x^2\right)^{9-r_3}\left(-\frac{1}{3x}\right)^{r_3}$$

$$= {}^9C_6\left(\frac{3}{2}\right)^3\left(-\frac{1}{3}\right)^6 + 2\ {}^9C_7\left(\frac{3}{2}\right)^2\left(-\frac{1}{3}\right)^7 = \frac{17}{54}$$

Hindi. $(1 + x + 2x^3)\left(\frac{3}{2}x^2 - \frac{1}{3x}\right)^9$ के प्रसार में x^0 का गुणांक

$\Rightarrow \left(\frac{3}{2}x^2 - \frac{1}{3x}\right)^9$ के प्रसार में x^0 का गुणांक $+\left(\frac{3}{2}x^2 - \frac{1}{3x}\right)^9$ के प्रसार में x^{-1} का गुणांक $+ 2\left(\frac{3}{2}x^2 - \frac{1}{3x}\right)^9$ के प्रसार में x^{-3} का गुणांक

$= {}^9C_{r_1}\left(\frac{3}{2}x^2\right)^{9-r_1}\left(-\frac{1}{3x}\right)^{r_1} + {}^9C_{r_2}\left(\frac{3}{2}x^2\right)^{9-r_2}\left(-\frac{1}{3x}\right)^{r_2} + 2\ {}^9C_{r_3}\left(\frac{3}{2}x^2\right)^{9-r_3}\left(-\frac{1}{3x}\right)^{r_3}$

$= {}^9C_6\left(\frac{3}{2}\right)^3\left(-\frac{1}{3}\right)^6 + 2\ {}^9C_7\left(\frac{3}{2}\right)^2\left(-\frac{1}{3}\right)^7 = \frac{17}{54}$

A-7. (i) Find the coefficient of x^5 in $(1 + 2x)^6(1 - x)^7$.
(ii) Find the coefficient of x^4 in $(1 + 2x)^4 (2 - x)^5$
(i) $(1 + 2x)^6(1 - x)^7$ में x^5 का गुणांक ज्ञात कीजिए।
(ii) $(1 + 2x)^4 (2 - x)^5$ में x^4 का गुणांक ज्ञात कीजिए।

Ans. (i) 171
(ii) –438

Sol. (i) $(1 + 2x)^6 (1 - x)^7$
$= (1 + {}^6C_1 (2x) + {}^6C_2 (2x)^2 + {}^6C_3 (2x)^3 + {}^6C_4 (2x)^4 + {}^6C_5 (2x)^5 + (2x)^6) (1 - x)^7$.
$= (1 + 12x + 60x^2 + 160 x^3 + 240x^4 + 192 x^5 + 64 x^6) (1 - x)^7$
$= 1 \times$ coeff of x^5 + 12 x coeff. of x^4 + 60 × coeff of x^3 + 160 × coeff of x^2 + 240 × coeff of x + 192 × constant term.
$= 1 \times x^5$ का गुणांक + 12 (x^4 का गुणांक) + 60 (x^3 का गुणांक) + 160 (x^2 का गुणांक) + 240 (x^2 का गुणांक) + 192 × अचर पद

$= 1 \times (-1)^5 . {}^7C_5 + 12 \times {}^7C_4 - 60 \times {}^7C_3 + 160 \times {}^7C_2 - 240 \times {}^7C_1 + 192 \times 1$
$= - 21 + 420 - 2100 + 3360 - 1680 + 192 = 171$

(ii) $(1 + 2x)^4 (2 - x)^5$
$[1 + {}^4C_1 (2x) + {}^4C_2 (2x)^2 + {}^4C_3 (2x)^3 + {}^4C_4(2x)^4] (2 -x)^5$
$= (1 + 8x + 24x^2 + 32x^3 + 16x^4) (2 - x)^5$
coefficient of x^4 = 1 × coefficient of x^4 + 8 × coefficient of x^3 + 24 × coefficient of x^2 + 32 × coefficient of x + 16 × constant term
x^4 का गुणांक = 1 × x^4 का गुणांक + 8 × x^3 का गुणांक + 24 × x^2 का गुणांक + 32 × x का गुणांक+ 16 × अचर पद
$T_{r+1} = {}^5C_r . 2^{5-r} (-x)^r = (-1)^r\ {}^5C_r\ 2^{5-r}\ x^r = 1 \times {}^5C_4 \times 2^1 - 8 \times {}^5C_3 \times 2^2 + 24 \times {}^5C_2 \times 2^3 - 32 \times {}^5C_1 \times 2^4 + 16\times2^5$

$= 10 - 320 + 1920 - 2560 + 512 = -438$

A-8. In the expansion of $\left(x^3 - \frac{1}{x^2}\right)^n$, $n \in N$, if the sum of the coefficients of x^5 and x^{10} is 0, then n is :

Ans. 15

यदि $\left(x^3 - \frac{1}{x^2}\right)^n$, $n \in N$ के प्रसार में x^5 और x^{10} के गुणांको का योग शून्य हो, तो n है–

Sol. $\left(x^3 - \frac{1}{x^2}\right)^n$

General term = $\frac{n!}{n!(n-r)!}\ (-1)^{n-r}\ x^{5r-2n}$

If 5r – 2n = 5, then 5r = 2n + 5 $\Rightarrow$ $r = \frac{2n}{5} + 1$

If 5r – 2n = 10, then 5r = 2n + 10 $\Rightarrow$ $r = \frac{2n}{5} + 2$

Let n = 5k

Now $\frac{5k!}{(2k+1)!(3k-1)!} - \frac{5k!}{(2k+2)!(3k-2)!} = 0$

$\Rightarrow$ $\frac{1}{3k-1} - \frac{1}{2k+2} = 0$ $\Rightarrow$ k = 3 $\Rightarrow$ n = 15

Hindi. $\left(x^3 - \frac{1}{x^2}\right)^n$

व्यापक पद = $\frac{n!}{n!(n-r)!}$ $(-1)^{n-r}\, x^{5r-2n}$

यदि 5r – 2n = 5, तब 5r = 2n + 5 $\Rightarrow$ $r = \frac{2n}{5} + 1$

यदि 5r – 2n = 10, तब 5r = 2n + 10 $\Rightarrow$ $r = \frac{2n}{5} + 2$

माना n = 5k

अब $\frac{5k!}{(2k+1)!(3k-1)!} - \frac{5k!}{(2k+2)!(3k-2)!} = 0$

$\Rightarrow$ $\frac{1}{3k-1} - \frac{1}{2k+2} = 0$ $\Rightarrow$ k = 3 $\Rightarrow$ n = 15

Section (B) : Middle term, Remainder & Numerically/Algebrically Greatest terms
खण्ड (B) : मध्य पद, शेषफल और संख्यात्मक/बीजगणितीय महत्तम पद

B-1. Find the middle term(s) in the expansion of
निम्न के प्रसार में मध्य पद ज्ञात करो–

(i) $\left(\frac{x}{y} - \frac{y}{x}\right)^7$ (ii) $(1 - 2x + x^2)^n$

Ans. (i) $-\frac{35x}{y}, \frac{35y}{x}$ (ii) $(-1)^n \frac{(2n)!}{n!\ n!} x^n$

Sol. **(i)** $\left(\frac{x}{y} - \frac{y}{x}\right)^7$ T_4 & T_5 are the middle term

(ii) $(1 - 2x + x^2)^n = (x - 1)^{2n}$
$T_{n+1} = {}^{2n}C_n (-1)^n x^n$

Hindi. **(i)** $\left(\frac{x}{y} - \frac{y}{x}\right)^7$ के प्रसार में T_4 और T_5 मध्य पद है

(ii) $(1 - 2x + x^2)^n = (x - 1)^{2n}$
$T_{n+1} = {}^{2n}C_n (-1)^n x^n$

B-2. Prove that the co-efficient of the middle term in the expansion of $(1 + x)^{2n}$ is equal to the sum of the co-efficients of middle terms in the expansion of $(1 + x)^{2n-1}$.
सिद्ध करो कि $(1 + x)^{2n}$ के प्रसार में मध्य पद का गुणांक, $(1 + x)^{2n-1}$ के प्रसार में मध्य पदों के गुणांकों के योगफल के बराबर है।

Sol. Co-efficient of middle term $(1 + x)^{2n} = {}^{2n}C_n$

${}^{2n}C_n = {}^{2n-1}C_{n-1} + {}^{2n-1}C_n$

Hindi. $(1 + x)^{2n}$ के प्रसार में मध्य पद का गुणांक $= {}^{2n}C_n$

${}^{2n}C_n = {}^{2n-1}C_{n-1} + {}^{2n-1}C_n$

B-3. (i) Find the remainder when 7^{98} is divided by 5

(ii) Using binomial theorem prove that $6^n - 5n$ always leaves the remainder 1 when divided by 25.

(iii) Find the last digit, last two digits and last three digits of the number $(27)^{27}$.

(i) यदि 7^{98} को 5 से विभाजित किया जाए, तो शेषफल ज्ञात करो।

(ii) द्विपद प्रमेय का उपयोग करते हुए सिद्ध कीजिए कि $6^n - 5n$ को 25 से विभाजित करने पर प्राप्त शेषफल सदैव 1 होता है।

(iii) $(27)^{27}$ का अन्तिम अंक, अन्तिम दो अंक व अन्तिम तीन अंक ज्ञात करो।

Ans. (i) 4

(iii) 3, 03, 803

Sol. (i) $7^{98} = (50 - 1)^{49} = {}^{49}C_0(50)^{49} - {}^{49}C_1(50)^{48} + \ldots\ldots - {}^{49}C_{49}$ $\Rightarrow$ Remainder शेषफल $= 5 - 1 = 4$

(ii) $6^n - 5n = (5 + 1)^n - 5n = 5^n + {}^nC_1.5^{n-1} + \ldots\ldots + {}^nC_{n-2}\,.\,5^2 + {}^nC_{n-1}\,.\,5 + 1 - 5n = 25\lambda + 1$

(iii) $(27)^{27} = 3^{81} = 3.(9)^{40}$

$= 3(10 - 1)^{40} = 3(10^{40} - {}^{40}C_1.10^{39} + \ldots\ldots + {}^{40}C_{38}.10^2 - {}^{40}C_{39}.10 + 1) = 3(1000\lambda - 400 + 1)$

Last 3 digits of this number = 803.

इस संख्या के अन्तिम 3 अंक = 803.

B-4. Which is larger : $(99^{50} + 100^{50})$ or $(101)^{50}$.

$(99^{50} + 100^{50})$ तथा $(101)^{50}$ में से कौनसा बड़ा है ?

Ans. 101^{50}

Sol. $(100 + 1)^{50} - 100^{50} - (100 - 1)^{50} = 2[{}^{50}C_1(100)^{49} + {}^{50}C_3(100)^{47} + \ldots\ldots + {}^{50}C_{49}(100)] - (100)^{50} > 0$

$(101)^{50} > (99^{50} + 100^{50})$

B-5. (i) Find numerically greatest term(s) in the expansion of $(3 - 5x)^{15}$ when $x = \frac{1}{5}$

(ii) Which term is the numerically greatest term in the expansion of $(2x + 5y)^{34}$, when $x = 3$ & $y = 2$?

(i) यदि $x = \frac{1}{5}$ तब $(3 - 5x)^{15}$ के प्रसार में महत्तम संख्यात्मक मान वाला(वाले) पद ज्ञात करो।

(ii) $(2x + 5y)^{34}$ के विस्तार में संख्यात्मक महत्तम पद होगा जब $x = 3$ तथा $y = 2$?

Ans. **(i)** $T_4 = -455 \times 3^{12}$ and $T_5 = 455 \times 3^{12}$

(ii) 22

Sol **(i)** For numerically greatest term in $(x + a)^n$ $r = \left[\frac{n+1}{1+\left|\frac{x}{a}\right|}\right] = \left[\frac{15+1}{1+\left|\frac{3}{1}\right|}\right] = 4$

Since value of $\frac{n+1}{1+\left|\frac{x}{a}\right|}$ is itself an integer. There are two terms, whose numerical values are greatest

T_4 and T_5

$T_4 = {}^{15}C_3 (3)^{12} (-1)^3 = -455 \times 3^{12}$

$T_5 = {}^{15}C_4 (3)^{11} (-1)^4 = 455 \times 3^{12}$

(i) $(x + a)^n$ के प्रसार में महत्तम संख्यात्मक मान वाले पद के लिए $r = \left[\dfrac{n+1}{1+\left|\dfrac{x}{a}\right|}\right] = \left[\dfrac{15+1}{1+\left|\dfrac{3}{1}\right|}\right] = 4$

चूँकि $\dfrac{n+1}{1+\left|\dfrac{x}{a}\right|}$ का मान एक पूर्णांक है अतः यहाँ दो पद है जिनका संख्यात्मक मान महत्तम है।

T_4 तथा T_5

$T_4 = {}^{15}C_3 (3)^{12} (-1)^3 = -455 \times 3^{12}$

$T_5 = {}^{15}C_4 (3)^{11} (-1)^4 = 455 \times 3^{12}$

(ii) For numerically greatest term महत्तम संख्यात्मक मान वाले पद के लिये $r = \left[\dfrac{n+1}{1+\left|\dfrac{x}{a}\right|}\right] = \left[\dfrac{34+1}{1+\left|\dfrac{6}{10}\right|}\right] \Rightarrow r = 21.$

B-6. Find the term in the expansion of $(2x - 5)^6$ which have

(i) Greatest binomial coefficient
(ii) Greatest numerical coefficient
(iii) Algebrically greatest coefficient
(iv) Algebrically least coefficient

$(2x - 5)^6$ के प्रसार में वह पद ज्ञात करो जो रखता है

(i) महत्तम द्विपद गुणांक
(ii) महत्तम संख्यात्मक गुणांक
(iii) महत्तम बीजगणितीय गुणांक
(iv) न्यूनतम बीजगणितीय गुणांक

Ans. (i) T_4 (ii) T_5, T_6 (iii) T_5 (iv) T_6

Sol. $(2x - 5)^6$

(i) Greatest binomical Co-efficient is of middle term $= T_{\frac{6}{2}+1} = T_4$

(ii) For greatest numerical term $r = \left[\dfrac{6+1}{1+\left|\dfrac{2}{5}\right|}\right] = \left[\dfrac{35}{7}\right] = 5$

Since $\dfrac{n+1}{1+\left|\dfrac{x}{a}\right|}$ itself is an integer.

$\therefore$ T_5 and T_6 both terms have are greatest numerical value

(iii) The positive term of greatest numerical value is Algebraically greatest i.e. T_5.

(iv) The negative term of greatest numerical value is algebraically least i.e. T_6

Hindi. $(2x - 5)^6$

(i) महत्तम द्विपद गुणांक वाला पद मध्य पद होता है अतः $= T_{\frac{6}{2}+1} = T_4$

(ii) महत्तम संख्यात्मक मान वाला पद $r = \left[\dfrac{6+1}{1+\left|\dfrac{2}{5}\right|}\right] = \left[\dfrac{35}{7}\right] = 5$

चूँकि $\frac{n+1}{1+\left|\frac{x}{a}\right|}$ एक पूर्णांक है अतः

T_5 तथा T_6 दोनों पद महत्तम संख्यात्मक मान वाले पद होंगे

(iii) बीजगणितीय महत्तम मान वाला पद धनात्मक महत्तम संख्यात्मक मान वाले पद के बराबर होता है, अर्थात् T_5

(iv) बीजगणितीय न्यूनतम मान वाला पद ऋणात्मक महत्तम संख्यात्मक मान वाले पद के बराबर होता है, अर्थात् T_6

Section (C) : Summation of series, Variable upper index & Product of binomial coefficients

खण्ड (C) : श्रेणी का योग, चर ऊपरी सूचकांक एवं द्विपद गुणांको का गुणन

C-1. If $C_0, C_1, C_2, \ldots C_n$ are the binomial coefficients in the expansion of $(1 + x)^n$ then prove that :
यदि $\mathbf{(1 + x)^n}$, $\mathbf{n \in N}$ के प्रसार में $\mathbf{C_0, C_1, C_2,.., C_n}$ द्विपद गुणांक है, तो सिद्ध करो :

(i)_ $C_0 - \frac{C_1}{\sqrt{2}} + \frac{C_2}{2} - \frac{C_3}{2\sqrt{2}}$ …….. upto (n + 1) terms equal to $\left(1-\frac{1}{\sqrt{2}}\right)^n$

(ii)_. $-C_1(3)^{n-1}(\sqrt{5})^1 + C_2(3)^{n-2}5 - C_3 3^{n-3}(5\sqrt{5})$ …..upto (n) terms equal to $(3-\sqrt{5})^n - 3^n$

(iii) $\frac{(3.2-1)}{2}C_1 + \frac{3^2.2^2-1}{2^2}C_2 + \frac{3^3.2^3-1}{2^3}C_3 + \ldots\ldots + \frac{3^n.2^n-1}{2^n}C_n = \frac{2^{3n}-3^n}{2^n}$

Sol. **(i)** Obvious

(ii) Obvious

(iii) $(C_0 + 3C_1 + 3^2C_2 + \ldots\ldots + 3^nC_n) - \left(C_0 + \frac{C_1}{2} + \frac{C_2}{2^2} + \ldots\ldots + \frac{C_n}{2^2}\right)$

$= 4^n - \left(\frac{3}{2}\right)^n = 2^{2n} - \frac{3^n}{2^n} = \frac{2^{3n}-3^n}{2^n}$

C-2. If $C_0, C_1, C_2, \ldots C_n$ are the binomial coefficients in the expansion of $(1 + x)^n$ then prove that :
यदि $\mathbf{(1 + x)^n}$, $\mathbf{n \in N}$ के प्रसार में $\mathbf{C_0, C_1, C_2,.., C_n}$ द्विपद गुणांक है, तो सिद्ध करो :

(i) $\frac{C_1}{C_0} + 2\frac{C_2}{C_1} + 3\frac{C_3}{C_2} + \ldots\ldots + n\frac{C_n}{C_{n-1}} = \frac{n(n+1)}{2}$

(ii) $(C_0 + C_1)(C_1 + C_2)(C_2 + C_3)(C_3 + C_4)\ldots\ldots(C_{n-1} + C_n) = \frac{C_0C_1C_2\ldots\ldots C_{n-1}(n+1)^n}{n!}$.

(iii) $C_o - 2C_1 + 3C_2 - 4C_3 + \ldots. + (-1)^n(n+1)C_n = 0$

(iv) $4C_0 + \frac{4^2}{2}.C_1 + \frac{4^3}{3}C_2 + \ldots\ldots\ldots\ldots + \frac{4^{n+1}}{n+1}C_n = \frac{5^{n+1}-1}{n+1}$

(v) $\frac{2^2.C_0}{1\,.\,2} + \frac{2^3.C_1}{2\,.\,3} + \frac{2^4.C_2}{3\,.\,4} + \ldots\ldots + \frac{2^{n+2}.C_n}{(n+1)(n+2)} = \frac{3^{n+2}-2n-5}{(n+1)(n+2)}$

(vi) $2.C_o + \frac{2^2.C_1}{2} + \frac{2^3.C_2}{3} + \frac{2^4.C_3}{4} + \ldots\ldots \frac{2^{n+1}.C_n}{n+1} = \frac{3^{n+1}-1}{n+1}$

Sol. **(i)** $\frac{C_1}{C_0}+2\frac{C_2}{C_1}+3\frac{C_3}{C_2}+$ ....

$= 1 + 2 + 3 + ...+ (n - 1) + n = \frac{n(n+1)}{2}$

(ii) $(C_0 + C_1)(C_1 + C_2)..........(C_{n-1} + C_n)$

$= ({}^{n+1}C_1)({}^{n+1}C_2)........({}^{n+1}C_n) = ((n + 1).C_0)\left(\frac{n+1}{2}.C_1\right)\,.......\,\left(\frac{n+1}{n}.C_{n-1}\right)$

$= \frac{(n+1)^n}{n\;!}C_0C_1........C_{n-1}$

(iii) $(1 + x)^n = C_0 + C_1x + ...+ C_nx^n$

$x(1 + x)^n = C_0x + C_1x^2 + ...+ C_nx^{n+1}$

Differentiating w.r.t. x x के सापेक्ष अवकलन करने पर

$(1 + x)^n + n\,x\,(1 + x)^{n-1} = C_0 + 2C_1x + ... + (n + 1)\,C_nx^n$

Putting $x = -1$ रखने पर

$C_0 - 2C_1 + ...+ (-1)^n (n + 1)\,C_n = 0$

(iv) $S = \sum_{r=0}^{n}\frac{{}^nC_r}{r+1}4^{r+1} = \frac{1}{n+1}\sum_{r=0}^{n}{}^{n+1}C_{r+1}\;\;4^{r+1} = \frac{1}{n+1}\,[C_14 + C_24^2 + ...\, C_{n+1}4^{n+1}] = \frac{1}{n+1}\;\;[5^{n+1} - 1]$

Aliter

$(1 + x)^n = {}^nC_0 + {}^nC_1x + ... + {}^nC_n\,x^n$

$\frac{(1+x)^{n+1}}{n+1}-\frac{1}{n+1} = {}^nC_0\,x + \frac{{}^nC_1x^2}{2} + ...\;\frac{{}^nC_nx^{n+1}}{n+1}$

put x = 4

then $C_0\,.\,4 + \frac{4^2}{2}C_1 + ...+ \frac{4^{n+1}}{n+1}C_n = \frac{5^{n+1}-1}{n+1}$

(iv) $S = \sum_{r=0}^{n}\frac{{}^nC_r}{r+1}4^{r+1} = \frac{1}{n+1}\sum_{r=0}^{n}{}^{n+1}C_{r+1}\;\;4^{r+1} = \frac{1}{n+1}\;\;[C_14 + C_24^2 + ...\, C_{n+1}4^{n+1}]$

$= \frac{1}{n+1}\,[5^{n+1} - 1]$

वैकल्पिक :

$(1 + x)^n = {}^nC_0 + {}^nC_1x + ... + {}^nC_n\,x^n$

$\frac{(1+x)^{n+1}}{n+1}-\frac{1}{n+1} = {}^nC_0\,x + \frac{{}^nC_1x^2}{2} + ...\;\frac{{}^nC_nx^{n+1}}{n+1}$

x = 4 रखने पर

तब $C_0\,.\,4 + \frac{4^2}{2}\;C_1 + ...+ \frac{4^{n+1}}{n+1}C_n = \frac{5^{n+1}-1}{n+1}$

(v) $(1+x)^n = C_0 + C_1x + ...+ C_nx^n$

Integrating from 0 to x

$\frac{(1+x)^{n+1}-1}{n+1} = C_0x + \frac{C_1x^2}{2} +........+ \frac{C_nx^{n+1}}{n+1}$

Again integrate 0 to x

$\frac{(1+x)^{n+2}-1}{(n+1)(n+2)} - \frac{x}{n+1} = \frac{C_0x^2}{2} + \frac{C_1x^3}{2.3} + ... + \frac{C_nx^{n+1}}{(n+1)(n+2)}$

put x = 2

$$\frac{2^2.C_0}{2}+\frac{2^3.C_1}{2.3}+......+\frac{2^{n+2}C_n}{(n+1)(n+2)}=\frac{3^{n+2}-2n-5}{(n+1)(n+2)}$$

(v) $(1+x)^n = C_0 + C_1x + ...+ C_nx^n$

0 से x तक समाकलन करने पर

$$\frac{(1+x)^{n+1}-1}{n+1} = C_0x + \frac{C_1x^2}{2} +........+\frac{C_nx^{n+1}}{n+1}$$

पुनः 0 से x तक समाकलन करने पर

$$\frac{(1+x)^{n+2}-1}{(n+1)(n+2)}-\frac{x}{n+1}=\frac{C_0x^2}{2}+\frac{C_1x^3}{2.3}+.......+\frac{C_nx^{n+1}}{(n+1)(n+2)}$$

x = 2 रखने पर

$$\frac{2^2.C_0}{2}+\frac{2^3.C_1}{2.3}+......+\frac{2^{n+2}C_n}{(n+1)(n+2)}=\frac{3^{n+2}-2n-5}{(n+1)(n+2)}$$

(vi) $(1 + x)^n = {}^nC_0 + {}^nC_1\, x + ...+ {}^nC_nx^n$

$$\int_0^2 (1+x)^n dx = C_0x + \frac{C_1x^2}{2} + ... + \frac{C_nx^{n+1}}{n+1}\Bigg|_0^2$$

$$\frac{3^{n+1}-1}{n+1} = 2.C_0 + \frac{2^2C_1}{2} + ... + \frac{2^{n+1}C_n}{n+1}$$

C-3. Prove that सिद्ध करो

(i) ${}^nC_r + {}^{n-1}C_r + {}^{n-2}C_r + ... + {}^rC_r = {}^{n+1}C_{r+1}$

(ii) ${}^{10}C_2 + {}^{11}C_2 + {}^{12}C_2 + ...+ {}^{19}C_2 = 1020$

Sol. **(i)** ${}^nC_r + {}^{n-1}C_r + ... + {}^rC_r$ = Co-efficient of x^r in $(1 + x)^n + (1 + x)^{n-1} + ... + (1 + x)^r$

= Co-efficient of x^r in $(1 + x)^r \left[\frac{(1+x)^{n-r+1}-1}{x}\right]$ = Co-efficient of x^{r+1} in $(1 + x)^{n+1} = {}^{n+1}C_{r+1}$

Hindi. **(i)** ${}^nC_r + {}^{n-1}C_r + ... + {}^rC_r = (1 + x)^n + (1 + x)^{n-1} + ... + (1 + x)^r$ के प्रसार में x^r का गुणांक

$= (1 + x)^r \left[\frac{(1+x)^{n-r+1}-1}{x}\right]$ के प्रसार में x^r का गुणांक

$= (1 + x)^{n+1}$ के प्रसार में x^r का गुणांक $= {}^{n+1}C_{r+1}$

(ii) ${}^{20}C_3 - {}^{10}C_3 = \frac{20\times19\times18}{3\times2\times1}-\frac{10\times9\times8}{3\times2\times1} = 120 = 1140 - 120 = 1020$

C-4. If $(1 + x)^n = C_0 + C_1\, x + C_2\, x^2 + ... + C_n\, x^n$, prove that

यदि $(1 + x)^n = C_0 + C_1\, x + C_2\, x^2 + ... + C_n\, x^n$, सिद्ध करो

(i) $C_0\, C_3 + C_1\, C_4 + ... + C_{n-3}\, C_n = \frac{(2n)!}{(n+3)\ !\ (n-3)!}$

(ii) $C_0\, C_r + C_1\, C_{r+1} + ... + C_{n-r}\, C_n = \frac{(2n)!}{(n+r)\ !\ (n-r)!}$

(iii) $C_0^2 - C_1^2 + C_2^2 - C_3^2 + ... + (-1)^n\, C_n^2 = 0$ or $(-1)^{n/2}\, C_{n/2}$ according as n is odd or even.

$C_0^2 - C_1^2 + C_2^2 - C_3^2 +...... + (-1)^n\, C_n^2 = 0$ या $(-1)^{n/2}\, C_{n/2}$ यदि n विषम या सम है।

[Revision Planner]

Sol. (i) $(1 + x)^n = C_0 + C_1 x + ... + C_n x^n$

$(x + 1)^n = C_0x^n + C_1 x^{n-1} + ... + C_n$

$C_0C_3 + C_1C_4 + ...+ C_{n-3}C_n$

$=$ Co-efficient of x^{n-3} in $(1 + x)^{2n} = {}^{2n}C_{n-3}$

(ii) $C_0C_r + C_1C_{r+1} + ...+ C_{n-r}C_n$

$=$ co-efficient of x^{n-r} in $(1+x)^{2n}$

$= {}^{2n}C_{n-r}$

(iii) $(1 + x)^n = C_0 + C_1 x + ... + C_n x^n$

$(x - 1)^n = C_0x^n - C_1 x^{n-1} + ... +(-1)\ {}^nC_n$

$C_0^2 - C_1^2 + ...+(-1)^nC_n^2$

$=$ co-efficient of x^n in $(x^2 -1)^n = 0$ if n is odd

$= {}^nC_{n/2}(-1)^{n/2}$ if n is even

Hindi. (i) $(1 + x)^n = C_0 + C_1 x + ... + C_n x^n$

$(x + 1)^n = C_0x^n + C_1 x^{n-1} + ... + C_n$

$C_0C_3 + C_1C_4 + ...+ C_{n-3}C_n$

$= (1 + x)^{2n}$ में x^{n-3} का गुणांक $= {}^{2n}C_{n-3}$

(ii) $C_0C_r + C_1C_{r+1} + ...+ C_{n-r}C_n$

$= (1 + x)^{2n}$ में x^{n-r} का गुणांक

$= {}^{2n}C_{n-r}$

(iii) $(1 + x)^n = C_0 + C_1 x + ... + C_n x^n$

$(x - 1)^n = C_0x^n - C_1 x^{n-1} + ... +(-1)\ {}^nC_n$

$C_0^2 - C_1^2 + ...+(-1)^nC_n^2$

$= (x^2 -1)^n$ में x^n का गुणांक $= 0$ यदि n विषम है।

$= {}^nC_{n/2}(-1)^{n/2}$ यदि n सम है।

Section (D) : Negative & fractional index, Multinomial theorem
खण्ड (D) : ऋणात्मक व भिन्न घातांक, बहुपदीय प्रमेय

D-1. Find the co-efficient of x^6 in the expansion of $(1 - 2x)^{-5/2}$.

$(1 - 2x)^{-5/2}$ के प्रसार में x^6 का गुणांक ज्ञात कीजिए।

Ans. $\dfrac{15015}{16}$

Sol. In the expansion of $(1 - 2x)^{-5/2}$

$$T_{r+1} = \frac{\frac{5}{2}\left(\frac{5}{2}+1\right)\left(\frac{5}{2}+2\right).......\left(\frac{5}{2}+r-1\right)}{r\ !} \cdot (-1)^r (2)^r x^r$$

$$\therefore \text{ Coefficient of } x^6 = \frac{\frac{5}{2}\left(\frac{5}{2}+1\right)\left(\frac{5}{2}+2\right)\left(\frac{5}{2}+3\right)\left(\frac{5}{2}+4\right)\left(\frac{5}{2}+5\right)}{6\ !} = \frac{15015}{16}$$

Hindi. $(1 - 2x)^{-5/2}$ के प्रसार में

$$T_{r+1} = \frac{\frac{5}{2}\left(\frac{5}{2}+1\right)\left(\frac{5}{2}+2\right).......\left(\frac{5}{2}+r-1\right)}{r\ !} \cdot (-1)^r (2)^r x^r$$

$$\therefore\ x^6 \text{ का गुणांक } = \frac{\frac{5}{2}\left(\frac{5}{2}+1\right)\left(\frac{5}{2}+2\right)\left(\frac{5}{2}+3\right)\left(\frac{5}{2}+4\right)\left(\frac{5}{2}+5\right)}{6\ !} = \frac{15015}{16}$$

D-2. (i) Find the coefficient of x^{12} in $\dfrac{4+2x-x^2}{(1+x)^3}$

(ii) Find the coefficient of x^{100} in $\dfrac{3-5x}{(1-x)^2}$

(i) $\dfrac{4+2x-x^2}{(1+x)^3}$ में x^{12} का गुणांक ज्ञात कीजिए।

(ii) $\dfrac{3-5x}{(1-x)^2}$ में x^{100} का गुणांक ज्ञात कीजिए।

Ans. (i) 142
(ii) – 197

Sol. (i) $(4 + 2x - x^2)\,(1 + x)^{-3}$
$= 4 \times$ coeff of $x^{12} + 2 \times$ coeff of $x^{11} - 1 \times$ coeff of x^{10}
$= 4$ (x^{12} का गुणांक) $+ 2$ (x^{11} का गुणांक) $- 1$ (x^{10} का गुणांक)

In $(1 + x)^{-3}$
$T_{r+1} = (-1)^r\ {}^{3+r-1}C_r x^r$
$= (-1)^r\ {}^{r+2}C_r\, x^r$
$= 4 \times {}^{14}C_{12} - 2 \times {}^{13}C_{11} - {}^{12}C_{10}$
$= 4 \times \dfrac{14\times13}{2} - 2x\dfrac{13\times12}{2} - \dfrac{12\times11}{2}$
$= 364 - 156 - 66 = 142$

(ii) $(3 - 5x)\,(1 - x)^{-2}$
$= 3 \times$ coeff of $x^{100} - 5 \times$ coeff of x^{99}
$= 3$ (x^{100} का गुणांक) $- 5$ (x^{99} का गुणांक)
In $(1 - x)^{-2}$
$T_{r+1} = {}^{2+r-1}C_r\, x^r = {}^{r+1}C_r\, x^r$
$= 3 \times {}^{101}C_{100} - 5 \times {}^{100}C_{99}$
$= 3 \times 101 - 5 \times 100$
$= -197$

D-3. Assuming ' x ' to be so small that x^2 and higher powers of ' x ' can be neglected, show that, $\dfrac{\left(1+\frac{3}{4}x\right)^{-4}(16-3x)^{1/2}}{(8+x)^{2/3}}$ is approximately equal to, $1 - \dfrac{305}{96}x$.

यदि 'x' का मान इतना अल्प है कि x^2 और 'x' की उच्च घातों को नगण्य माना जा सकता है तो प्रदर्शित कीजिए कि $\dfrac{\left(1+\frac{3}{4}x\right)^{-4}(16-3x)^{1/2}}{(8+x)^{2/3}}$ का मान लगभग $1 - \dfrac{305}{96}x$ है।

Sol. $$\frac{\left(1+\frac{3}{4}x\right)^{-4}(16-3x)^{1/2}}{(8+x)^{2/3}} = \frac{(1-3x)\ .\ 4\left(1-\frac{3}{32}x\right)}{4\left(1+\frac{2x}{24}\right)} = \left(1-3x-\frac{3}{32}x\right)\left(1-\frac{x}{12}\right)$$

$$= 1 - \frac{x}{12} - 3x - \frac{3}{32}x = 1 - \frac{305}{96}x$$

D-4. (i) Find the coefficient of $a^5 b^4 c^7$ in the expansion of $(bc + ca + ab)^8$.
(ii) Sum of coefficients of odd powers of x in expansion of $(9x^2 + x - 8)^6$
(i) $(bc + ca + ab)^8$ के प्रसार में $a^5 b^4 c^7$ का गुणांक ज्ञात करो।
(ii) $(9x^2 + x - 8)^6$ के प्रसार में x की विषम घातों के गुणांको का योगफल है।
Ans. (i) 280 (ii) 2^5

Sol. (i) $(bc + ca + ab)^8$

$$\frac{8\,!}{r_1\,!\;r_2\,!\;r_3\,!}\,(bc)^{r_1}(ca)^{r_2}(ab)^{r_3}$$

$$\left.\begin{matrix} r_2 + r_3 = 5 \\ r_1 + r_3 = 4 \\ r_2 + r_1 = 7 \end{matrix}\right\} \Rightarrow \quad r_2 = 4, \quad r_1 = 3, \quad r_3 = 1$$

or या $\frac{8!}{4!3!1!} = 280$

(ii) $(9x^2 + x - 8)^6 = a_0 + a_1x + a_2x^2 + \ldots\ldots a_{12}x^{12}$

$2^6 = a_0 + a_1 + \ldots\ldots + a_{12} \quad (x = 1)$

$0 = a_0 - a_1 + \ldots\ldots. + a_{12} \quad (x = -1)$

$\Rightarrow \quad a_1 + a_3 + \ldots\ldots. + a_{11} = 2^5$

D-5. Find the coefficient of x^7 in $(1 - 2x + x^3)^5$.
$(1 - 2x + x^3)^5$ में x^7 का गुणांक ज्ञात कीजिए।
Ans. 20

Sol. Co-efficient of x^7 in $(1 - 2x + x^3)^5$ के प्रसार में x^7 का गुणांक है

$$= \frac{n\,!}{r_1!r_2!r_3!}(1)^{r_1}(-2x)^{r_2}(x^3)^{r_3} \;\; = r_2 + 3r_3 = 7 \;\&\; r_1, r_2, r_3 \le 5$$

(i) $r_2 = 4, r_3 = 1, r_1 = 0$ (ii) $r_2 = 1, r_3 = 2, r_1 = 2 = \frac{5!}{4!1!}(2)^4 + \frac{5!}{2!2!1!} \times (-2)^1 = 20$

PART - II : ONLY ONE OPTION CORRECT TYPE

भाग - II : केवल एक सही विकल्प प्रकार (ONLY ONE OPTION CORRECT TYPE)

Section (A) : General Term & Coefficient of x^k in $(ax + b)^n$
खण्ड (A) : व्यापक पद एवं $(ax + b)^n$ में x^k का गुणांक

A-1. The $(m + 1)^{th}$ term of $\left(\frac{x}{y} + \frac{y}{x}\right)^{2m+1}$ is:
(A) independent of x (B) a constant
(C*) depends on the ratio x/y and m (D) none of these

$\left(\frac{x}{y} + \frac{y}{x}\right)^{2m+1}$ का (m + 1)वाँ पद
(A) x पर निर्भर नहीं है। (B) अचर है।
(C) अनुपात x/y और m पर निर्भर है। (D) इनमें से कोई नहीं

Sol. $${}^{2m+1}C_m \left(\frac{x}{y}\right)^{m+1}\left(\frac{y}{x}\right)^m = {}^{2m+1}C_m\left(\frac{x}{y}\right)$$

Dependent upon the ratio $\frac{x}{y}$ and m. अनुपात x/y और m पर निर्भर है।

A-2. The total number of distinct terms in the expansion of, $(x + a)^{100} + (x - a)^{100}$ after simplification is :
(A) 50 (B) 202 (C*) 51 (D) none of these
$(x + a)^{100} + (x - a)^{100}$ के प्रसार में सरल करने के बाद भिन्न–भिन्न पदों की कुल संख्या है :
(A) 50 (B) 202 (C) 51 (D) इनमें से कोई नहीं

Sol. $(x + a)^{100} + (x - a)^{100}$

$= 2\left({}^{100}C_0 \ x^{100} + {}^{100}C_2 \ x^{98}a^2 + ... + {}^{100}C_{100} \ a^{100}\right)$

Number of terms = 51 terms (पदों की संख्या = 51 पद)

A-3. The value of, $\dfrac{18^3 + 7^3 + 3 \,.\, 18 \,.\, 7 \,.\, 25}{3^6 + 6 \,.\, 243 \,.\, 2 + 15 \,.\, 81 \,.\, 4 + 20 \,.\, 27 \,.\, 8 + 15 \,.\, 9 \,.\, 16 + 6 \,.\, 3 \,.\, 32 + 64}$ is :
(A*) 1 (B) 2 (C) 3 (D) none
$\dfrac{18^3 + 7^3 + 3 \,.\, 18 \,.\, 7 \,.\, 25}{3^6 + 6 \,.\, 243 \,.\, 2 + 15 \,.\, 81 \,.\, 4 + 20 \,.\, 27 \,.\, 8 + 15 \,.\, 9 \,.\, 16 + 6 \,.\, 3 \,.\, 32 + 64}$ का मान है –
(A) 1 (B) 2 (C) 3 (D) इनमें से कोई नहीं

Sol. $\dfrac{(18+7)^3}{(3+2)^6} = \dfrac{25^3}{5^6} = 1$

A-4. In the expansion of $\left(3 - \sqrt{\frac{17}{4} + 3\sqrt{2}}\right)^{15}$ the 11th term is a:
(A) positive integer (B*) positive irrational number
(C) negative integer (D) negative irrational number.
$\left(3 - \sqrt{\frac{17}{4} + 3\sqrt{2}}\right)^{15}$ के प्रसार में 11वाँ पद है –
(A) धनात्मक पूर्णांक (B) धनात्मक अपरिमेय संख्या
(C) ऋणात्मक पूर्णांक (D) ऋणात्मक अपरिमेय संख्या

Sol. $T_{11} = {}^{15}C_{10}\,(3)^5 \left(-\sqrt{\frac{17}{4} + 3\sqrt{2}}\right)^{10} = {}^{15}C_{10}(3)^5\left(\frac{17}{4} + 3\sqrt{2}\right)^5$ = a positive irrational number

Hindi $T_{11} = {}^{15}C_{10}\,(3)^5 \left(-\sqrt{\frac{17}{4} + 3\sqrt{2}}\right)^{10} = {}^{15}C_{10}(3)^5\left(\frac{17}{4} + 3\sqrt{2}\right)^5$ = धनात्मक अपरिमेय संख्या

A-5. If the second term of the expansion $\left[a^{1/13} + \frac{a}{\sqrt{a^{-1}}}\right]^n$ is $14a^{5/2}$, then the value of $\frac{{}^nC_3}{{}^nC_2}$ is:
यदि $\left[a^{1/13} + \frac{a}{\sqrt{a^{-1}}}\right]^n$ के प्रसार में द्वितीय पद $14a^{5/2}$ है, तो $\frac{{}^nC_3}{{}^nC_2}$ का मान है–
(A*) 4 (B) 3 (C) 12 (D) 6

Sol. $T_2 = {}^nC_1\,(a^{1/13})^{n-1}\,(a^{3/2}) = 14a^{5/2} \Rightarrow n = 14 \quad \therefore \quad \frac{{}^nC_3}{{}^nC_2} = 4$

A-6. In the expansion of $(7^{1/3} + 11^{1/9})^{6561}$, the number of terms free from radicals is:
$(7^{1/3} + 11^{1/9})^{6561}$ के प्रसार में करणी चिन्ह (radical sign) से रहित पदों की संख्या है –
(A*) 730 (B) 729 (C) 725 (D) 750

Sol. $T_{r+1} = {}^{6561}C_r (7)^{\frac{6561-r}{3}} (11^{1/9})^r$
Here r should be multiple of 9
r = 0, 9,18...............6561
Number of terms = 730

Hindi $T_{r+1} = {}^{6561}C_r (7)^{\frac{6561-r}{3}} (11^{1/9})^r$
यहाँ r, 9 का गुणज होना चाहिए
r = 0, 9,18...............6561
पदों की संख्या = 730

A-7. The value of m, for which the coefficients of the $(2m + 1)^{th}$ and $(4m + 5)^{th}$ terms in the expansion of $(1 + x)^{10}$ are equal, is
$(1 + x)^{10}$ के प्रसार में (2m + 1)वें एवं (4m + 5)वें पदों के गुणांक समान हैं, तो m का मान है–
(A) 3 (B*) 1 (C) 5 (D) 8

Sol. $\left.\begin{array}{l} T_{2m+1} \Rightarrow {}^{10}C_{2m} \\ T_{4m+5} \Rightarrow {}^{10}C_{4m+4} \end{array}\right]$ equal बराबर है।

$2m + 4m + 4 = 10 \Rightarrow 6m + 4 = 10$
$m = 1$

A-8. The co-efficient of x in the expansion of $(1 - 2x^3 + 3x^5)\left(1+\frac{1}{x}\right)^8$ is :

$(1 - 2x^3 + 3x^5)\left(1+\frac{1}{x}\right)^8$ के प्रसार में x का गुणांक है–
(A) 56 (B) 65 (C*) 154 (D) 62

Sol. $(1 - 2x^3 + 3x^5)\left(1+\frac{1}{x}\right)^8$

Co-efficient of x (x का गुणांक) $= -2.\,{}^8C_2 + 3.{}^8C_4 = 154$

A-9. Given that the term of the expansion $(x^{1/3} - x^{-1/2})^{15}$ which does not contain x is 5 m, where $m \in N$, then m=
(A) 1100 (B) 1010 (C*) 1001 (D) 1002
यदि $(x^{1/3} - x^{-1/2})^{15}$ के प्रसार में x से स्वतंत्र पद 5 m के बराबर है, जहाँ $m \in N$, तो m =
(A) 1100 (B) 1010 (C) 1001 (D) 1002

Sol. $(x^{1/3} - x^{-1/2})^{15}$

$T_{r+1} = {}^{15}C_r\ x^{\left(\frac{15-r}{3}\right)} (-x^{-1/2})^r$

For constant term $\frac{15-r}{3} - \frac{r}{2} = 0 \Rightarrow r = 6$

Co-efficient of $x^0 = {}^{15}C_6 = 5\,m \Rightarrow m = 1001$

Hindi. $(x^{1/3} - x^{-1/2})^{15}$

$T_{r+1} = {}^{15}C_r\ x^{\left(\frac{15-r}{3}\right)} (-x^{-1/2})^r$

अचर पद के लिये $\frac{15-r}{3} - \frac{r}{2} = 0 \Rightarrow r = 6$

x^0 का गुणांक $= {}^{15}C_6 = 5\,m \Rightarrow m = 1001$

A-10. The term independent of x in the expansion of $\left(x - \frac{1}{x}\right)^4 \left(x + \frac{1}{x}\right)^3$ is:

$\left(x - \frac{1}{x}\right)^4 \left(x + \frac{1}{x}\right)^3$ के प्रसार में x से स्वतंत्र पद हैं–

(A) – 3 (B*) 0 (C) 1 (D) 3

Sol. $\left(x-\frac{1}{x}\right)\left(x^2-\frac{1}{x^2}\right)^3 = \left(x-\frac{1}{x}\right)({}^3C_0 x^6 - {}^3C_1 x^2 + {}^3C_2 x^{-2} - {}^3C_3 x^{-6})$

There is no term independent of x यहाँ कोई भी पद x से स्वतंत्र नहीं है।

Section (B) : Middle term, Remainder & Numerically/Algebrically Greatest terms
खण्ड (B) : मध्य पद, शेषफल और संख्यात्मक/बीजगणितीय महत्तम पद

B-1. If $k \in R^+$ and the middle term of $\left(\frac{k}{2}+2\right)^8$ is 1120, then value of k is:

(A) 3 (B*) 2 (C) 1 (D) 4

यदि $k \in R^+$ और $\left(\frac{k}{2}+2\right)^8$ का मध्य पद 1120 है, तो k का मान होगा :

(A) 3 (B*) 2 (C) 1 (D) 4

Sol. middle term = T_5

$T_5 = T_{4+1} = {}^8C_4 . k^4 = 1120 \Rightarrow k = 2$

Hindi. मध्य पद = T_5

$T_5 = T_{4+1} = {}^8C_4 . k^4 = 1120 \Rightarrow k = 2$

B-2. The remainder when 2^{2003} is divided by 17 is :

यदि 2^{2003} को 17 से विभाजित किया जाता है, तो शेषफल होगा –

(A) 1 (B) 2 (C*) 8 (D) 7

Sol. $2^{2003} = 8.(16)^{500}$

$= 8\,(17–1)^{500}$ $\therefore$ Remainder शेषफल = 8

B-3. The last two digits of the number 3^{400} are:

संख्या 3^{400} के अन्तिम दो अंक है :

(A) 81 (B) 43 (C) 29 (D*) 01

Sol. $(81)^{100} = (80 + 1)^{100} = {}^{100}C_0\,(80)^{100} + \text{........} + {}^{100}C_{99}\,(80)^1 + 1$

Last two digits अन्तिम दो अंक = 01

B-4. The last three digits in 10 ! are :

10 ! के मान में अन्तिम तीन अंक है –

(A*) 800 (B) 700 (C) 500 (D) 600

Sol. Last two digits in 10! are 00 and third digit = 8

Hindi 10! के मान में अन्तिम दो अंक 00 है तथा तीसरा अंक = 8

B-5. The value of $\sum_{r=1}^{10} r \,.\, \frac{{}^nC_r}{{}^nC_{r-1}}$ is equal to

$\sum_{r=1}^{10} r \,.\, \frac{{}^nC_r}{{}^nC_{r-1}}$ का मान बराबर है–

(A*) 5 (2n – 9) (B) 10 n (C) 9 (n – 4) (D) n –2

Sol. $\sum_{r=1}^{10} r.\frac{^nC_r}{^nC_{r-1}} = \sum_{r=1}^{10} n-r+1 = (n + 1) \times 10 - \frac{10\times 11}{2} = 10n - 45$

B-6. $\sum_{r=0}^{n-1} \frac{^nC_r}{^nC_r + {^nC_{r+1}}} =$

(A*) $\frac{n}{2}$ (B) $\frac{n+1}{2}$ (C) $(n+1)\ \frac{n}{2}$ (D) $\frac{n\ (n-1)}{2\ (n+1)}$

Sol. $\sum_{r=0}^{n-1} \frac{^nC_r}{^nC_r + {^nC_{r+1}}} = \sum_{r=0}^{n-1} \frac{r+1}{n+1} = \frac{1}{n+1}\ [1 + 2 + ...+n] = \frac{1}{n+1} \times \frac{n(n+1)}{2} = \frac{n}{2}$

B-7. Find numerically greatest term in the expansion of $(2 + 3x)^9$, when $x = 3/2$.
$(2 + 3x)^9$ के प्रसार में $x = 3/2$ के लिए महत्तम संख्यात्मक मान वाला पद है –
(A*) $^9C_6 . 2^9 . (3/2)^{12}$ (B) $^9C_3 . 2^9 . (3/2)^6$ (C) $^9C_5 . 2^9 . (3/2)^{10}$ (D) $^9C_4 . 2^9 . (3/2)^8$

Sol. For numerically greatest term $r = \left[\frac{n+1}{1+\left|\frac{x}{a}\right|}\right] = \left[\frac{9+1}{1+\left|\frac{4}{9}\right|}\right]$ $\Rightarrow r = 6$

Numerically greatest term $T_{r+1} = {^9C_6}(2)^3\left(\frac{9}{2}\right)^6$

Hindi. महत्तम संख्यात्मक मान वाले पद के लिये $r = \left[\frac{n+1}{1+\left|\frac{x}{a}\right|}\right] = \left[\frac{9+1}{1+\left|\frac{4}{9}\right|}\right]$ $\Rightarrow r = 6$

महत्तम संख्यात्मक पद $T_{r+1} = {^9C_6}(2)^3\left(\frac{9}{2}\right)^6$

B-8. The greatest integer less than or equal to $(\sqrt{2} + 1)^6$ is
$(\sqrt{2} + 1)^6$ से कम या बराबर महत्तम पूर्णांक है–
(A) 196 (B*) 197 (C) 198 (D) 199

Sol. T_{22} is the numerically greatest term. T_{22} संख्यात्मक महत्तम पद है।

$(\sqrt{2} + 1)^6 = I + f$

$(\sqrt{2} - 1)^6 = f'$

$2[{^6C_0} + {^6C_2} . 2 + {^6C_4}\ (2)^2 + ...] = I + f + f'$

$f + f' = 1$ or $f' = 1 - f$

$I = 2\ [{^6C_0} + {^6C_2}.2 + {^6C_4}.4 + {^6C_6}.8] - 1$

$I = 2\ [1 + 30 + 60 + 8] - 1 = 197$

Section (C) : Summation of series, Variable upper index & Product of binomial coefficients
खण्ड (C) : श्रेणी का योग, चर ऊपरी सूचकांक एवं द्विपद गुणांको का गुणन

C-1. $\frac{^{11}C_0}{1} + \frac{^{11}C_1}{2} + \frac{^{11}C_2}{3} + \ldots\ldots + \frac{^{11}C_{10}}{11} =$

(A) $\frac{2^{11}-1}{11}$ (B*) $\frac{2^{11}-1}{6}$ (C) $\frac{3^{11}-1}{11}$ (D) $\frac{3^{11}-1}{6}$

Sol. $\frac{^{11}C_0}{1} + \frac{^{11}C_1}{2} + \frac{^{11}C_2}{3} + \ldots\ldots + \frac{^{11}C_{10}}{11}$

$= \left[\frac{12}{1} \cdot {}^{11}C_0 + \frac{12}{2} \cdot {}^{11}C_1 + \frac{12}{3} \cdot {}^{11}C_2 + \ldots.. + \frac{12}{11} \cdot {}^{11}C_{10}\right] = \frac{1}{12}\left[{}^{12}C_1 + {}^{12}C_2 + {}^{12}C_3 + \ldots\ldots. + {}^{12}C_{11}\right]$

$= \frac{1}{12}(2^{12} - 2) = \frac{2^{11}-1}{6}$

C-2. The value of $\frac{C_0}{1.3} - \frac{C_1}{2.3} + \frac{C_2}{3.3} - \frac{C_3}{4.3} + \ldots\ldots\ldots + (-1)^n \frac{C_n}{(n+1)\ .\ 3}$ is :

(A) $\frac{3}{n+1}$ (B) $\frac{n+1}{3}$ (C*) $\frac{1}{3(n+1)}$ (D) none of these

$\frac{C_0}{1.3} - \frac{C_1}{2.3} + \frac{C_2}{3.3} - \frac{C_3}{4.3} + \ldots\ldots\ldots + (-1)^n \frac{C_n}{(n+1)\ .\ 3}$ का मान होगा–

(A) $\frac{3}{n+1}$ (B) $\frac{n+1}{3}$ (C*) $\frac{1}{3(n+1)}$ (D) इनमें से कोई नहीं

Sol. $\int_0^1 (1-x)^n dx = \int_0^1 \left(C_0 - C_1x + C_2x^2 - C_3x^3 + \ldots + (-1)^n C_nx^n\right)dx$

$\Rightarrow \frac{1}{n+1} = \left[C_0 - \frac{C_1}{2} + \frac{C_2}{3} - \ldots\ldots\ldots\ldots + (-1)^n \frac{C_n}{n+1}\right]$

$\Rightarrow \frac{1}{3}\left(\frac{1}{n+1}\right) = \frac{1}{3}\left[C_0 - \frac{C_1}{2} + \frac{C_2}{3} - \ldots\ldots\ldots\ldots\ldots\ldots\ldots + (-1)^n \frac{C_n}{n+1}\right]$

C-3. The value of the expression ${}^{47}C_4 + \sum_{j=1}^{5} {}^{52-j}C_3$ is equal to :

व्यंजक ${}^{47}C_4 + \sum_{j=1}^{5} {}^{52-j}C_3$ का मान बराबर है–

(A) ${}^{47}C_5$ (B) ${}^{52}C_5$ (C*) ${}^{52}C_4$ (D) ${}^{49}C_4$

Sol. ${}^{47}C_4 + {}^{51}C_3 + {}^{50}C_3 + {}^{49}C_3 + {}^{48}C_3 + {}^{47}C_3 = {}^{52}C_4$

C-4. The value of $\binom{50}{0}\binom{50}{1} + \binom{50}{1}\binom{50}{2} + \ldots\ldots\ldots.. + \binom{50}{49}\binom{50}{50}$ is, where ${}^nC_r = \binom{n}{r}$

$\binom{50}{0}\binom{50}{1} + \binom{50}{1}\binom{50}{2} + \ldots\ldots\ldots.. + \binom{50}{49}\binom{50}{50}$ का मान होगा, जहाँ ${}^nC_r =$

(A) $\binom{100}{50}$ (B*) $\binom{100}{51}$ (C) $\binom{50}{25}$ (D) $\binom{50}{25}^2$

Sol. ${}^{50}C_0 \times {}^{50}C_1 + {}^{50}C_1 \times {}^{50}C_2 + \ldots\ldots\ldots.. + {}^{50}C_{49} \times {}^{50}C_{50}$

$= {}^{50}C_0 \times {}^{50}C_{49} + {}^{50}C_1 \times {}^{50}C_{48} + \ldots\ldots\ldots + {}^{50}C_{49} \times {}^{50}C_0$

$=$ co-eff. of x^{49} in $(1 + x)^{100}$ $= {}^{100}C_{49}$

Hindi ${}^{50}C_0 \times {}^{50}C_1 + {}^{50}C_1 \times {}^{50}C_2 + \ldots\ldots\ldots + {}^{50}C_{49} \times {}^{50}C_{50}$

$= {}^{50}C_0 \times {}^{50}C_{49} + {}^{50}C_1 \times {}^{50}C_{48} + \ldots\ldots\ldots + {}^{50}C_{49} \times {}^{50}C_0$

$= (1 + x)^{100}$ में x^{49} का गुणांक $= {}^{100}C_{49}$

Section (D) : Negative & fractional index, Multinomial theorem
खण्ड (D) : ऋणात्मक व भिन्न घातांक, बहुपदीय प्रमेय

D-1. If $|x| < 1$, then the co-efficient of x^n in the expansion of $(1 + x + x^2 + x^3 + \ldots\ldots.)^2$ is

यदि $|x| < 1$, तो $(1 + x + x^2 + x^3 + \ldots\ldots.)^2$ के प्रसार में x^n का गुणांक है –

(A) n (B) n – 1 (C) n + 2 (D*) n + 1

Sol. Co-efficient of x^n in $(1 - x)^{-2} = {}^{2+n-1}C_1 = n + 1$

Hindi. $(1 - x)^{-2}$ के प्रसार में x^n का गुणांक $= {}^{2+n-1}C_1 = n + 1$

D-2. The co–efficient of x^4 in the expansion of $(1 - x + 2x^2)^{12}$ is:

$(1 - x + 2x^2)^{12}$ के प्रसार में x^4 का गुणांक है –

(A) ${}^{12}C_3$ (B) ${}^{13}C_3$ (C) ${}^{14}C_4$ (D*) ${}^{12}C_3 + 3\ {}^{13}C_3 + {}^{14}C_4$

Sol. $(1 - x + 2x^2)^{12}$

General term $= \dfrac{12!}{r_1!\ r_2!\ r_3!}(1)^{r_1}(-x)^{r_2}\left(2x^2\right)^{r_3}$

$r_2 + 2r_3 = 4 \Rightarrow$ $r_3 = 0, r_2 = 4, r_1 = 8$

$r_3 = 1, r_2 = 2, r_1 = 9$

$r_3 = 2, r_2 = 0, r_1 = 10$

Co-efficient of $x^4 = \dfrac{12!}{4!\ 8!} + \dfrac{12!}{2!\ 10!}(2)^2 + \dfrac{12!}{2!\ 9!} \times (2)$

$= {}^{12}C_8 + 4.\ {}^{12}C_{10} + 6\ .\ {}^{12}C_9$

$= {}^{12}C_3 + 3\ .\ {}^{13}C_3 + {}^{14}C_4$ (after solving)

Hindi. $(1 - x + 2x^2)^{12}$

व्यापक पद $= \dfrac{12!}{r_1!\ r_2!\ r_3!}(1)^{r_1}(-x)^{r_2}\left(2x^2\right)^{r_3}$

$r_2 + 2r_3 = 4 \Rightarrow$ $r_3 = 0, r_2 = 4, r_1 = 8$

$r_3 = 1, r_2 = 2, r_1 = 9$

$r_3 = 2, r_2 = 0, r_1 = 10$

x^4 का गुणांक $= \dfrac{12!}{4!\ 8!} + \dfrac{12!}{2!\ 10!}(2)^2 + \dfrac{12!}{2!\ 9!} \times (2)$

$= {}^{12}C_8 + 4.\ {}^{12}C_{10} + 6\ .\ {}^{12}C_9 = {}^{12}C_3 + 3\ .\ {}^{13}C_3 + {}^{14}C_4$ हल करने पर

D-3. If $(1 + x)^{10} = a_0 + a_1x + a_2x^2 + \ldots\ldots + a_{10}x^{10}$, then value of

$(a_0 - a_2 + a_4 - a_6 + a_8 - a_{10})^2 + (a_1 - a_3 + a_5 - a_7 + a_9)^2$ is

(A*) 2^{10} (B) 2 (C) 2^{20} (D) None of these

यदि $(1 + x)^{10} = a_0 + a_1x + a_2x^2 + \ldots\ldots + a_{10}x^{10}$ हो, तो $(a_0 - a_2 + a_4 - a_6 + a_8 - a_{10})^2 + (a_1 - a_3 + a_5 - a_7 + a_9)^2$ का मान है–

(A) 2^{10} (B) 2 (C) 2^{20} (D) इनमें से कोई नहीं

Sol. $(1 + x)^{10} = a_0 + a_1 + a_2x^2 + ... + a_{10}x^{10}$

Put $x = i$,

$(1 + i)^{10} = a_0 - a_2 + a_4 + ...+ a_{10} + i (a_1 - a_3 + ...+ a_9)$

$a_0 - a_2 + a_4 + ...+ a_{10}$ = real part of $(1 + i)^{10} = 2^5\cos 10\pi/4$

$a_1 - a_3 + ...$ = imaginary part of $(1 + i)^{10} = 2^5 \sin 10\pi/4$ ....(2)

$(1)^2 + (2)^2 = 2^{10}$

Hindi. $(1 + x)^{10} = a_0 + a_1 + a_2x^2 + ... + a_{10}x^{10}$

$x = i$ रखने पर

$(1 + i)^{10} = a_0 - a_2 + a_4 + ...+ a_{10} + i (a_1 - a_3 + ...+ a_9)$

$a_0 - a_2 + a_4 + ...+ a_{10} = (1 + i)^{10}$ का वास्तविक भाग $= 2^5\cos 10\pi/4$

$a_1 - a_3 + ... = (1 + i)^{10}$ का काल्पनिक भाग $= 2^5 \sin 10\pi/4$ ....(2)

$(1)^2 + (2)^2 = 2^{10}$

PART - III : MATCH THE COLUMN

भाग - III : कॉलम को सुमेलित कीजिए (MATCH THE COLUMN)

1.

	Column – I		**Column – II**
(A)	If $(r + 1)^{th}$ term is the first negative term in the expansion of $(1 + x)^{7/2}$, then the value of r (where $0 < x < 1$) is	(p)	divisible by 2
(B)	If the sum of the co-efficients in the expansion of $(1 + 2x)^n$ is 6561, and T_r is the greatest term in the expansion for $x = 1/2$ then r is	(q)	divisible by 5
(C)	${}^nC_r$ is divisible by n, $(1 < r < n)$ if n is	(r)	divisible by 10
(D)	The coefficient of x^4 in the expression $(1 + 2x + 3x^2 + 4x^3 + ...\text{up to } \infty)^{1/2}$ is c, $(c \in N)$, then c + 1 (where $\mid x \mid < 1$) is	(s)	a prime number

Ans. (A) $\rightarrow$ (q, s), (B) $\rightarrow$ (q,s), (C) $\rightarrow$ (s), (D) $\rightarrow$ (p, s)

	स्तम्भ – I		**स्तम्भ – II**
(A)	यदि $(1 + x)^{7/2}$ के प्रसार में $(r + 1)$वाँ पद प्रथम ऋणात्मक पद है, तो r का मान है – (जहाँ $0 < x < 1$)	(p)	2 से विभाजित है
(B)	यदि $(1 + 2x)^n$ के प्रसार में गुणांको का योगफल 6561, और $x = 1/2$ के लिए T_r अधिकतम पद है तब r है।	(q)	5 से विभाजित है
(C)	${}^nC_r$, $(1 < r < n)$, n से विभाजित होगा यदि n है,	(r)	10 से विभाजित है
(D)	व्यंजक $(1 + 2x + 3x^2 + 4x^3 + ...$ अनन्त पदों तक$)^{1/2}$ में x^4 का गुणांक c, $(c \in N)$ है, तो c + 1 है – (जहाँ $\mid x \mid < 1$)	(s)	एक अभाज्य संख्या

Sol. (A) We have, $T_{r+1} = \dfrac{\frac{7}{2}\left(\frac{7}{2}-1\right)\left(\frac{7}{2}-2\right)\ .....\ \left(\frac{7}{2}-r+1\right)\ x^r}{r\ !}$

This will be the first negative term when $\frac{7}{2} - r + 1 < 0$ i.e. $r > \frac{9}{2}$

Hence r = 5.

(B) $3^n = 6561$ (put x = 1) $\Rightarrow$ n = 8

$\dfrac{T_{r+1}}{T_r} = \dfrac{8-r+1}{r} \geq 1 \quad \Rightarrow \quad 8 - r + 1 \geq r \quad \Rightarrow \quad r \leq \dfrac{9}{2} \Rightarrow r = 4$ (5^{th} term is greatest)

(C) Obviously a prime number.

(D) We have : $(1 + 2x + 3x^2 + 4x^3 + ...)^{1/2}$

$= [(1 - x)^{-2}]^{1/2} = (1 - x)^{-1} = 1 + x + x^2 + ... + x^n + ...$

Hence, coefficient of $x^4 = 1 \quad \therefore \quad c = 1$, hence c + 1 = 2

Hindi (A) चूँकि $T_{r+1} = \dfrac{\frac{7}{2}\left(\frac{7}{2}-1\right)\left(\frac{7}{2}-2\right)\ .....\ \left(\frac{7}{2}-r+1\right)\ x^r}{r\ !}$

यह प्रथम ऋणात्मक पद होगा यदि $\frac{7}{2} - r + 1 < 0$ i.e. $r > \frac{9}{2}$

अतः r = 5.

(B) $3^n = 6561$ (x = 1 रखने पर) $\Rightarrow$ n = 8

$\dfrac{T_{r+1}}{T_r} = \dfrac{8-r+1}{r} \geq 1 \quad \Rightarrow \quad 8 - r + 1 \geq r \Rightarrow r \leq \Rightarrow r = 4$ (5th पद महत्तम है।)

(C) स्पष्टतयाः एक अभाज्य संख्या।

(D) $(1 + 2x + 3x^2 + 4x^3 + ...)^{1/2}$

$= [(1 - x)^{-2}]^{1/2} = (1 - x)^{-1} = 1 + x + x^2 + ... + x^n + ...$

अतः x^4 का गुणांक = 1 $\therefore$ c = 1, अतः c + 1 = 2

Exercise-2

Marked questions are recommended for Revision.
चिन्हित प्रश्न दोहराने योग्य प्रश्न है।

PART - I : ONLY ONE OPTION CORRECT TYPE

भाग-I : केवल एक सही विकल्प प्रकार (ONLY ONE OPTION CORRECT TYPE)

1. In the expansion of

$\left(3\sqrt{\frac{a}{b}}+3\sqrt{\frac{b}{\sqrt{a}}}\right)^{21}$, the term containing same powers of a & b is

(A) 11^{th} term (B*) 13^{th} term (C) 12^{th} term (D) 6^{th} term

$\left(3\sqrt{\frac{a}{b}}+3\sqrt{\frac{b}{\sqrt{a}}}\right)^{21}$, के विस्तार में a और b की समान घातों का पद है –

(A) 11^{th} वां पद (B*) 13^{th} वां पद (C) 12^{th} वां पद (D) 6^{th} वां पद

Sol. $T_{r+1} = {}^{21}C_r \left(\frac{a}{b}\right)^{\frac{21-r}{3}} \left(\frac{b}{\sqrt{a}}\right)^{\frac{r}{3}}$

$= {}^{21}C_r .\ a^{\frac{21-r}{3}-\frac{r}{6}}\ b^{\frac{r}{3}-\frac{21-r}{3}}$

$= {}^{21}C_r .\ a^{\frac{42-3r}{6}} .\ b^{\frac{2r-21}{3}}$

$= {}^{21}C_r .\ a^{\frac{14-r}{2}} .\ b^{\frac{2r-21}{3}}$

$\frac{14-r}{2} = \frac{2r-21}{3}$

$42 - 3r = 4r - 42$

$2r = 84$

$r = 12 \Rightarrow T_{13}$ term (T_{13}वां पद)

2. Consider the following statements :

S_1 : Number of dissimilar terms in the expansion of $(1 + x + x^2 + x^3)^n$ is $3n + 1$

S_2 : $(1 + x)(1 + x + x^2)(1 + x + x^2 + x^3)$...... $(1 + x + x^2 +...... + x^{100})$ when written in the ascending power of x then the highest exponent of x is 5000.

S_3 : $\sum_{k=1}^{n-r} {}^{n-k}C_r = {}^nC_{r+1}$

S_4 : If $(1 + x + x^2)^n = a_0 + a_1x + a_2x^2 + ... + a_{2n}x^{2n}$, then $a_0 + a_2 + a_4 + ... + a_{2n} = \frac{3^n-1}{2}$

State, in order, whether S_1, S_2, S_3, S_4 are true or false

माना कि निम्न कथन है –

S_1 : $(1 + x + x^2 + x^3)^n$ के प्रसार में असमान पदों की संख्या $3n + 1$ है।

S_2 : $(1 + x)(1 + x + x^2)(1 + x + x^2 + x^3)$...... $(1 + x + x^2 +...... + x^{100})$ को यदि x की बढ़ती हुई घातों के क्रम में लिखा जाता है, तो x की अधिकतम घात 5000 होगी –

S_3 : $\sum_{k=1}^{n-r} {}^{n-k}C_r = {}^nC_{r+1}$

S_4 : यदि $(1 + x + x^2)^n = a_0 + a_1x + a_2 x^2 + ... + a_{2n} x^{2n}$, तो $a_0 + a_2 + a_4 + ... + a_{2n} = \frac{3^n - 1}{2}$

S_1, S_2, S_3, S_4 के सत्य (T) या असत्य (F) होने का सही क्रम है –

(A*) TFTF (B) TTTT (C) FFFF (D) FTFT

Sol. S1 : Number of dissimilar terms will be same as in $(1 + x)^{3n}$ i.e. 3n + 1

S_2 : $(1 + x)(1 + x + x^2)...... + (1 + x + ... + x^{100})$
Highest exponent of x = 1 + 2 + ...+ 100
= 5050

Hindi. $(1 + x)(1 + x + x^2)...... + (1 + x + ... + x^{100})$
x की अधिकतम घात = 1 + 2 + ...+ 100
= 5050

S_3 : $\sum_{k=1}^{n-r} {}^{n-k}C_r = {}^xC_y \quad \Rightarrow {}^xC_y = {}^{n-1}C_r + {}^{n-2}C_r + ... {}^rC_r$

$\Rightarrow {}^xC_y$ = co-efficient of x^r in $((1+x)^r + ... + (1+x)^{n-1})$

= co-efficient of x^r in $(1+x)^r \left[\frac{(1+x)^{n-r} - 1}{x}\right]$ = co-efficient of x^{r+1} in $(1+x)^n = {}^nC_{r+1}$

Hindi $\sum_{k=1}^{n-r} {}^{n-k}C_r = {}^xC_y \quad \Rightarrow {}^xC_y = {}^{n-1}C_r + {}^{n-2}C_r + ... {}^rC_r$

$\Rightarrow {}^xC_y = ((1+x)^r + ... + (1+x)^{n-1})$ में x^r का गुणांक

$= (1+x)^r \left[\frac{(1+x)^{n-r} - 1}{x}\right]$ में x^r का गुणांक = $(1+x)^n$ में x^{r+1} का गुणांक = ${}^nC_{r+1}$

S_4 : $(1 + x + x^2)^n = a_0 + a_1x + a_2x^2 + ...+ a_{2n}x^{2n}$
put x = 1 रखने पर
$3^n = a_0 + a_1 + a_2 + ...+ a_{2n}$ ....(i)
$x = -1$
$1 = a_0 - a_1 + a_2 + ...+ a_{2n}$ ....(ii)
adding (i) & (ii) (i) तथा (ii) को जोड़ने पर
$\frac{3^n + 1}{2} = a_0 + a_2 + ...+ a_{2n.}$

3. .If $\frac{{}^nC_r + 4\ {}^nC_{r+1} + 6\ {}^nC_{r+2} + 4\ {}^nC_{r+3} + {}^nC_{r+4}}{{}^nC_r + 3\,{}^nC_{r+1} + 3\ {}^nC_{r+2} + {}^nC_{r+3}} = \frac{n+k}{r+k}$ then the value of k is :

यदि $\frac{{}^nC_r + 4\ {}^nC_{r+1} + 6\ {}^nC_{r+2} + 4\ {}^nC_{r+3} + {}^nC_{r+4}}{{}^nC_r + 3\,{}^nC_{r+1} + 3\ {}^nC_{r+2} + {}^nC_{r+3}} = \frac{n+k}{r+k}$ हो तो k का मान ज्ञात कीजिए।

(A) 1 (B) 2 (C*) 4 (D) 5

Sol. Numerator = ${}^nC_r + {}^nC_{r+1} + 3({}^nC_{r+1} + {}^nC_{r+2}) + 3({}^nC_{r+2} + {}^nC_{r+3}) + {}^nC_{r+3} + {}^nC_{r+4}$
$= {}^{n+1}C_{r+1} + 3\ {}^{n+1}C_{r+2} + 3\ {}^{n+1}C_{r+3} + {}^{n+1}C_{r+4}$
$= {}^{n+1}C_{r+1} + {}^{n+1}C_{r+2} + 2({}^{n+1}C_{r+2} + {}^{n+1}C_{r+3}) + {}^{n+1}C_{r+3} + {}^{n+1}C_{r+4}$
$= {}^{n+2}C_{r+2} + 2\ {}^{n+2}C_{r+3} + {}^{n+2}C_{r+4}$

$= {}^{n+2}C_{r+2} + {}^{n+2}C_{r+3} + {}^{n+2}C_{r+4} + {}^{n+2}C_{r+3}$
$= {}^{n+3}C_{r+3} + {}^{n+3}C_{r+4} = {}^{n+4}C_{r+4}$
Denominator $= {}^{n}C_{r} + {}^{n}C_{r+1} + 2\,({}^{n}C_{r+1} + {}^{n}C_{r+2}) + ({}^{n}C_{r+2} + {}^{n}C_{r+3})$
$= {}^{n+1}C_{r+1} + 2\,{}^{n+1}C_{r+2} + {}^{n+1}C_{r+3}$
$= {}^{n+1}C_{r+1} + {}^{n+1}C_{r+2} + {}^{n+1}C_{r+2} + {}^{n+1}C_{r+3}$
$= {}^{n+2}C_{r+2} + {}^{n+2}C_{r+3} = {}^{n+3}C_{r+3}$

$\therefore$ The expression is equal to : $\dfrac{{}^{n+4}C_{r+4}}{{}^{n+3}C_{r+3}} = \dfrac{(n+4)!(r+3)!(n-r)!}{(r+4)!(n-r)!(n+3)!} = \dfrac{n+4}{r+4}$

$\therefore$ k = 4

4. The co-efficient of x^5 in the expansion of $(1 + x)^{21} + (1 + x)^{22} + \ldots\ldots + (1 + x)^{30}$ is :
$(1 + x)^{21} + (1 + x)^{22} + \ldots\ldots + (1 + x)^{30}$ के प्रसार में x^5 का गुणांक है :
(A) ${}^{51}C_5$ (B) ${}^{9}C_5$ (C*) ${}^{31}C_6 - {}^{21}C_6$ (D) ${}^{30}C_5 + {}^{20}C_5$

Sol. $(1 + x)^{21}\,[1 + (1 + x) + \ldots\ldots (1 + x)^9] = (1 + x)^{21}\left[\dfrac{(1+x)^{10}-1}{x}\right] = \dfrac{(1+x)^{31}-(1+x)^{21}}{x}$

Coefficient of $x^5 = {}^{31}C_6 - {}^{21}C_6$
x^5 का गुणांक $= {}^{31}C_6 - {}^{21}C_6$

5. The coefficient of x^{52} in the expansion $\sum\limits_{m=0}^{100} {}^{100}C_m\,(x-3)^{100-m}.\,2^m$ is :

$\sum\limits_{m=0}^{100} {}^{100}C_m\,(x-3)^{100-m}.\,2^m$ के प्रसार में x^{52} का गुणांक है –
(A) ${}^{100}C_{47}$ (B*) ${}^{100}C_{48}$ (C) $-{}^{100}C_{52}$ (D) $-{}^{100}C_{100}$

Sol. $S = \sum\limits_{m=0}^{100} {}^{100}C_m (x-3)^{100-m}2^m$

$S = {}^{100}C_0\,(x-3)^{100} + {}^{100}C_1\,(x-3)^{99}\,.\,2 + \ldots\ldots + {}^{100}C_{100}\,.\,2^{100}$
$S = (2 + (x-3))^{100} = (x-1)^{100}$
Co-efficient of $x^{52} = {}^{100}C_{52} = {}^{100}C_{48}$
x^{52} का गुणांक $= {}^{100}C_{52} = {}^{100}C_{48}$

6. The sum of the coefficients of all the integral powers of x in the expansion of $\left(1+2\sqrt{x}\right)^{40}$ is :

$\left(1+2\sqrt{x}\right)^{40}$ के प्रसार में x की सभी पूर्णांक घातों के गुणांकों का योगफल है –

(A) $3^{40} + 1$ (B) $3^{40} - 1$ (C) $\dfrac{1}{2}\,(3^{40} - 1)$ (D*) $\dfrac{1}{2}\,(3^{40} + 1)$

Sol. $\left(1+2\sqrt{x}\right)^{40} = {}^{40}C_0 + {}^{40}C_1\,2\sqrt{x} + \ldots\ldots + {}^{40}C_{40}\,(2\sqrt{x})^{40}$

$(1-2\sqrt{x})^{40} = {}^{40}C_0 - {}^{40}C_1\,2\sqrt{x} + \ldots\ldots + {}^{40}C_{40}\,(2\sqrt{x})^{40}$

$(1+2\sqrt{x})^{40} + (1-2\sqrt{x})^{40} = 2\,[{}^{40}C_0 + {}^{40}C_2\,(2\sqrt{x})^2 + \ldots\ldots + {}^{40}C_{40}\,(2\sqrt{x})^{40}]$

Putting x = 1 रखने पर

${}^{40}C_0 + {}^{40}C_2(2)^2 + \ldots\ldots + {}^{40}C_{40}\,(2)^{40} = \dfrac{3^{40}+1}{2}$

7. $\sum\limits_{r=0}^{n}(-1)^r\,{}^{n}C_r.\dfrac{(1+r\ell n10)}{(1+\ell n10^n)^r} =$

$\sum_{r=0}^{n}(-1)^r\ {}^nC_r.\frac{(1+r\ell n10)}{(1+\ell n10^n)^r}$ का मान है–

(A*) 0 (B) 1/2 (C) 1 (D) None of these (इनमें से कोई)

Sol. $\sum_{r=0}^{n}(-1)^r\ {}^nC_r.\frac{1}{(1+n\ell n10)^r}+\sum_{r=0}^{n}(-1)^r r.{}^nC_r\ \frac{\ell n10}{(1+n\ell n10)^r}$

$= \left(1-\frac{1}{1+n\ell n10}\right)^n + n\ell n10\sum_{r=1}^{n}(-1)^r\ {}^{n-1}C_{r-1}\frac{1}{(1+n\ell n10)^r}$

$= \left(\frac{n\ell 10}{1+n\ell 10}\right)^n - \frac{n\ell 10}{1+\ell 10}\sum_{r=1}^{n}(-1)^{r-1}\ {}^{n-1}C_{r-1}\left(\frac{1}{n\ell n10}\right)^{r-1}$

$= \left(\frac{n\ell 10}{1+n\ell 10}\right)^n - \frac{n\ell 10}{1+n\ell 10}\times\left(1-\frac{1}{1+n\ell 10}\right)^{n-1}$

$= \left(\frac{n\ell 10}{1+\ell n10}\right)^n - \frac{n\ell 10}{1+\ell n\ell 10}\times\frac{(n\ell 10)^{n-1}}{(1+n\ell 10)^{n-1}}=0$

8. The coefficient of the term independent of x in the expansion of $\left(\frac{x+1}{x^{\frac{2}{3}}-x^{\frac{1}{3}}+1}-\frac{x-1}{x-x^{\frac{1}{2}}}\right)^{10}$ is :

$\left(\frac{x+1}{x^{\frac{2}{3}}-x^{\frac{1}{3}}+1}-\frac{x-1}{x-x^{\frac{1}{2}}}\right)^{10}$ के प्रसार में x से स्वतंत्र पद का गुणांक है –

(A) 70 (B) 112 (C) 105 (D*) 210

Sol. $\left(\frac{x+1}{x^{\frac{2}{3}}-x^{\frac{1}{3}}+1}-\frac{x-1}{x-x^{\frac{1}{2}}}\right)^{10} = \left(x^{1/3}+1-1-\frac{1}{\sqrt{x}}\right)^{10}$

$T_{r+1} = {}^{10}C_r(x^{1/3})^{10-r}\left(-\frac{1}{\sqrt{x}}\right)^r$

For independent term $\frac{10-r}{3}-\frac{r}{2}=0 \Rightarrow r = 4$

Coefficient of the term independent of x = ${}^{10}C_4$

Hindi. $\left(\frac{x+1}{x^{\frac{2}{3}}-x^{\frac{1}{3}}+1}-\frac{x-1}{x-x^{\frac{1}{2}}}\right)^{10} = \left(x^{1/3}+1-1-\frac{1}{\sqrt{x}}\right)^{10}$

$T_{r+1} = {}^{10}C_r(x^{1/3})^{10-r}\left(-\frac{1}{\sqrt{x}}\right)^r$

स्वतन्त्र पद हेतु $\frac{10-r}{3}-\frac{r}{2}=0 \Rightarrow r = 4$

x से स्वतन्त्र पद का गुणांक = ${}^{10}C_4$

9. Coefficient of x^{n-1} in the expansion of, $(x + 3)^n + (x + 3)^{n-1}(x + 2) + (x + 3)^{n-2}(x + 2)^2 + ... + (x + 2)^n$ is :

$(x + 3)^n + (x + 3)^{n-1}(x + 2) + (x + 3)^{n-2}(x + 2)^2 + ... + (x + 2)^n$ के प्रसार में x^{n-1} का गुणांक है –

(A) ${}^{n+1}C_2(3)$ (B) ${}^{n-1}C_2(5)$ (C*) ${}^{n+1}C_2(5)$ (D) ${}^{n}C_2(5)$

Sol. $(x + 3)^n + (x + 3)^{n-1} (x + 2) + ... + (x + 2)^n = (x + 3)^n \left(\dfrac{1-\left(\dfrac{x+2}{x+3}\right)^{n+1}}{1-\dfrac{x+2}{x+3}} \right) = [(x + 3)^{n+1} - (x + 2)^{n+1}]$

Coefficient of x^{n-1} का गुणांक $= {}^{n+1}C_{n-1} (3)^2 - {}^{n+1}C_{n-1} \times 4$

10. Let $f(n) = 10^n + 3.4^{n+2} + 5$, $n \in N$. The greatest value of the integer which divides f(n) for all n is :
(A) 27 (B*) 9 (C) 3 (D) None of these
माना $f(n) = 10^n + 3.4^{n+2} + 5$, $n \in N$ तब वह अधिकतम पूर्णांक जो f(n) को n के प्रत्येक मान के लिए विभाजित करता है–
(A) 27 (B) 9 (C) 3 (D) इनमें से कोई नहीं

Sol. $f(n) = 10^n + 3.4^{n+2} + 5$
put n = 1
f(1) = 10 + 192 + 5 = 207 this is divisible by 3 and 9
f(1) = 10 + 192 + 5 = 207 यह 3 तथा 9 दोनों से भाज्य है।

11. If $(1 + x)^n = \sum_{r=0}^{n} a_r x^r$ and $b_r = 1 + \dfrac{a_r}{a_{r-1}}$ and $\prod_{r=1}^{n} b_r = \dfrac{(101)^{100}}{100!}$, then n equals to :
(A) 99 (B*) 100 (C) 101 (D) 102
यदि $(1 + x)^n = \sum_{r=0}^{n} a_r x^r$ और $b_r = 1 + \dfrac{a_r}{a_{r-1}}$ और $\prod_{r=1}^{n} b_r = \dfrac{(101)^{100}}{100!}$, तो n बराबर है :
(A) 99 (B) 100 (C) 101 (D) 102

Sol. $(1 + x)^n = \sum_{r=0}^{n} a_r x^r = a_0 + a_1x + ...+ a_nx^n$

$b_r = 1 + \dfrac{a_r}{a_{r-1}} = 1 + \dfrac{n-r+1}{r} = \dfrac{n+1}{r}$

$\prod_{n=1}^{n} b_r = b_1b_2........b_n = \dfrac{(n+1)^n}{1.\ \ 2.3......n} = \dfrac{(101)^{100}}{100!} \quad \Rightarrow n = 100$

12. Number of rational terms in the expansion of $\left(1+\sqrt{2}+\sqrt{5}\right)^6$ is :
$\left(1+\sqrt{2}+\sqrt{5}\right)^6$ के विस्तार में परिमेंय पदों की संख्या है –
(A) 7 (B*) 10 (C) 6 (D) 8

Sol. General term is = व्यापक पद

$= \dfrac{6!}{r_1!\ r_2!\ r_3!}(1)^{r_1}(\sqrt{2})^{r_2}(\sqrt{5})^{r_3} = \dfrac{6!}{r_1!\ r_2!\ r_3!}(2)^{\frac{r_2}{2}}(2)^{\frac{r_3}{2}}$ where जहाँ $r_1 + r_2 + r_3 = 6$

r_1	r_2	r_3
2	4	0
4	2	0
0	4	2
2	2	2
4	0	2
0	2	4
2	0	4
0	0	6
0	6	0
6	0	0

10 terms are possible

r_1	r_2	r_3
2	4	0
4	2	0
0	4	2
2	2	2
4	0	2
0	2	4
2	0	4
0	0	6
0	6	0
6	0	0

10 पद संभव है।

13. If $S = {}^{404}C_4 - {}^4C_1 . {}^{303}C_4 + {}^4C_2 . {}^{202}C_4 - {}^4C_3 . {}^{101}C_4 = (101)^k$ then k equals to :

यदि $S = {}^{404}C_4 - {}^4C_1 . {}^{303}C_4 + {}^4C_2 . {}^{202}C_4 - {}^4C_3 . {}^{101}C_4 = (101)^k$ तब k का मान है–

(A) 1 (B) 2 (C*) 4 (D) 6

Sol. S = coeff. of x^4 in S = x^4 में का गुणांक

$= \left[{}^4C_0 . \left((1+x)^{101}\right)^4 - {}^4C_1\left((1+x)^{101}\right)^3 + {}^4C_2\left((1+x)^{101}\right)^2 - {}^4C_3\left((1+x)^{101}\right)^1 + {}^4C_4 \right] - {}^4C_4$

$= \left((1+x)^{101} - 1\right)^4 - 1$

$= (1 + {}^{101}C_1\, x + {}^{101}C_2\, x^2 + {}^{101}C_3\, x^3 + {}^{101}C_4\, x^4 + \dots\dots\dots - 1)^4 - 1$

$= x^4\left({}^{101}C_1 + {}^{101}C_2 x + {}^{101}C_3 x^2 + \dots.\right)^4 - 1$

$= x^4\left(101 + ({}^{101}C_2 x + {}^{101}C_3 x^2 + \dots.)\right)^4 - 1$

$= 101^4$

k = 4

14. ${}^{10}C_0^{\,2} - {}^{10}C_1^{\,2} + {}^{10}C_2^{\,2} - \dots. - \left({}^{10}C_9\right)^2 + \left({}^{10}C_{10}\right)^2 =$

(A) 0 (B) $\left({}^{10}C_5\right)^2$ (C*) $-{}^{10}C_5$ (D) $2\,{}^9C_5$

Sol. $(1+x)^{10} = {}^{10}C_0 + {}^{10}C_1 x + {}^{10}C_2 x^2 + \dots + {}^{10}C_9 x^9 + {}^{10}C_{10} x^{10}$

$(x-1)^{10} = {}^{10}C_0 x^{10} - {}^{10}C_1 x^9 + {}^{10}C_2 x^8 - \dots - {}^{10}C_9 . x + {}^{10}C_{10}$

S = coff. of x^{10} in $\left(x^2 - 1\right)^{10}$

$= -{}^{10}C_5$

15. The sum $\sum_{r=0}^{n}$ (r + 1) C_r^2 is equal to :

योगफल $\sum_{r=0}^{n}$ (r + 1) C_r^2 बराबर है –

(A*) $\dfrac{(n+2)\,(2n-1)\,!}{n\,!\,(n-1)\,!}$ (B) $\dfrac{(n+2)\,(2n+1)\,!}{n\,!\,(n-1)\,!}$ (C) $\dfrac{(n+2)\,(2n+1)\,!}{n\,!\,(n+1)!}$ (D) $\dfrac{(n+2)\,(2n-1)\,!}{n\,!\,(n+1)\,!}$

Sol. $\because$ $(1+x)^n = C_0 + C_1 x + \dots\dots\dots\dots + C_n x^n$

Multiply by x & then differentiate

$(1+x)^n + x.\, n(1+x)^{n-1} = C_0 + 2C_1 x + \dots\dots\dots\dots + (n+1)C_n x^n$ ....(i)

and $(x+1)^n = C_0 x^n + C_1 x^{n-1} + \dots\dots\dots\dots + C_n$ ....(ii)

Multiply (i) & (ii) & equate the coefficient of x^n on both side

$C_0^2 + 2C_1^2 + \ldots\ldots.. + (n+1)\, C_n^2 = {}^{2n}C_n + n.\, {}^{2n-1}C_{n-1} = \dfrac{(2n)\,!}{(n!)^2} + n\dfrac{(2n-1)\,!}{n!(n-1)!} = (n+2)\dfrac{(2n-1)\,!}{n!(n-1)!}$

Hindi $\because$ $(1+x)^n = C_0 + C_1x + \ldots\ldots\ldots\ldots + C_nx^n$

x से गुणा करके अवकलन करने पर

$(1+x)^n + x.\, n(1+x)^{n-1} = C_0 + 2C_1x + \ldots\ldots\ldots\ldots + (n+1)C_nx^n$ ....(i)

तथा $(x+1)^n = C_0x^n + C_1x^{n-1} + \ldots\ldots\ldots\ldots + C_n$ ....(ii)

(i) और (ii) का गुणा करके दोनों पक्षों में x^n के गुणांकों की तुलना करने पर

$C_0^2 + 2C_1^2 + \ldots\ldots.. + (n+1)\, C_n^2 = {}^{2n}C_n + n.\, {}^{2n-1}C_{n-1} = \dfrac{(2n)\,!}{(n!)^2} + n\dfrac{(2n-1)\,!}{n!(n-1)!} = (n+2)\dfrac{(2n-1)\,!}{n!(n-1)!}$

16. If $(1 + x + x^2 + x^3)^5 = a_0 + a_1x + a_2x^2 + \ldots\ldots\ldots\ldots\ldots\ldots\ldots.. + a_{15}x^{15}$, then a_{10} equals to :

यदि $(1 + x + x^2 + x^3)^5 = a_0 + a_1x + a_2x^2 + \ldots\ldots\ldots\ldots\ldots\ldots\ldots.. + a_{15}x^{15}$ हो, तो a_{10} =

(A) 99 (B*) 101 (C) 100 (D) 110

Sol. $(x^4 - 1)^5\, (x-1)^{-5} = {}^5C_0\, (x-1)^{-5} - {}^5C_1\, x^4\, (x-1)^{-5} + {}^5C_2\, x^8\, (x-1)^{-5} = {}^5C_0 \times {}^{14}C_4 - {}^5C_1 \times {}^{10}C_6 + {}^5C_2 \times {}^6C_2 = 101$

17. If $a_n = \sum\limits_{r=0}^{n} \dfrac{1}{{}^nC_r}$, the value of $\sum\limits_{r=0}^{n} \dfrac{n-2r}{{}^nC_r}$ is :

यदि $a_n = \sum\limits_{r=0}^{n} \dfrac{1}{{}^nC_r}$ हो, तो $\sum\limits_{r=0}^{n} \dfrac{n-2r}{{}^nC_r}$ का मान होगा –

(A) $\dfrac{n}{2}a_n$ (B) $\dfrac{1}{4}a_n$ (C) na_n (D*) 0

Sol. $a_n = \sum\limits_{r=0}^{n} \dfrac{1}{{}^nC_r}$; $S = \sum\limits_{r=0}^{n} \dfrac{n-2r}{{}^nC_r}$; $S = \sum\limits_{r=0}^{n} \dfrac{n-2(n-r)}{{}^nC_r}$; $2S = 0 \Rightarrow S = 0$

18. The sum of: $3.{}^nC_0 - 8.{}^nC_1 + 13.{}^nC_2 - 18.{}^nC_3 + \ldots.$ upto (n+1) terms is $(n \geq 2)$:

(A*) zero (B) 1 (C) 2 (D) none of these

$3.{}^nC_0 - 8.{}^nC_1 + 13.{}^nC_2 - 18.{}^nC_3 + \ldots.$ के (n+1) पदों का योगफल है $(n \geq 2)$:

(A) शून्य (B) 1 (C) 2 (D) इनमें से कोई नहीं

Sol. $3\,.\,{}^nC_0 - 8\,.\,{}^nC_1 + 13\,.\,{}^nC_2 - 18\,{}^nC_3 + \ldots.$ up to (n + 1) terms

$(1 + x^5)^n = C_0 + C_1x^5 + C_2x^{10} + \ldots\ldots + C_nx^{5n}$

Multiplying by x^3 and differentiating w.r.t. x

$x^3\,.\, n(1 + x^5)^{n-1}\,.\, 5x^4 + 3x^2\,(1 + x^5)^n = 3C_0\, x^2 + 8C_1\, x^7 + 13C_2\, x^{12} + \ldots\ldots + (5n+3)\, C_n\, x^{5n+2}$

Now put $x = -1$

$3C_0 - 8C_1 + 13\, C_2 + \ldots\ldots\ldots. + (n+1)$ terms $= 0$

Hindi. $3\,.\,{}^nC_0 - 8\,.\,{}^nC_1 + 13\,.\,{}^nC_2 - 18\,{}^nC_3 + \ldots.$ (n + 1) पदों तक

$(1 + x^5)^n = C_0 + C_1x^5 + C_2x^{10} + \ldots\ldots + C_nx^{5n}$

x^3 से गुणा करके x के सापेक्ष गुणा करने पर

$x^3\,.\, n(1 + x^5)^{n-1}\,.\, 5x^4 + 3x^2\,(1 + x^5)^n = 3C_0\, x^2 + 8C_1\, x^7 + 13C_2\, x^{12} + \ldots\ldots + (5n+3)\, C_n\, x^{5n+2}$

अब $x = -1$ रखने पर

$3C_0 - 8C_1 + 13\, C_2 + \ldots\ldots\ldots. + (n+1)$ पदों तक $= 0$

19. If यदि $\sum\limits_{r=0}^{n-1} \left(\dfrac{{}^nC_r}{{}^nC_r + {}^nC_{r+1}} \right)^3 = \dfrac{4}{5}$ then तब n =

(A*) 4 (B) 6 (C) 8 (D) None of these इनमें से कोइ नहीं

Sol. $\sum_{r=0}^{n-1}\left(\frac{{}^nC_r}{{}^nC_r+{}^nC_{r+1}}\right)^3=\frac{4}{5}$ $\Rightarrow\sum_{r=0}^{n-1}\left(\frac{\frac{n!}{r!(n-r!)}}{\frac{(n+1)!}{(r+1)!(n-r)!}}\right)^3=\frac{4}{5}$ $\Rightarrow\sum_{r=0}^{n-1}\left(\frac{r+1}{n+1}\right)^3=\frac{4}{5}$

$\Rightarrow \frac{1}{(n+1)^3}(1^3+2^3+3^3+\ldots\ldots\ldots+n^3)=4/5$

$\Rightarrow \frac{1}{(n+1)^3}\times\frac{n^2(n+1)^2}{4}=\frac{4}{5}\Rightarrow$ $5n^2-16n-16=0$ $\Rightarrow$ $n=4$

20. The number of terms in the expansion of $\left(x^2+1+\frac{1}{x^2}\right)^n$, $n\in N$, is :

$\left(x^2+1+\frac{1}{x^2}\right)^n$, $n\in N$ के प्रसार में पदों की संख्या है –

(A) 2n (B) 3n (C*) 2n + 1 (D) 3n + 1

Sol. $\left(\left(x+\frac{1}{x}\right)^2-1\right)^n={}^nC_0\left(x+\frac{1}{x}\right)^{2n}-{}^nC_1\left(x+\frac{1}{x}\right)^{2n-2}+\ldots.+{}^nC_n(-1)^n$

Total number of terms = 2n + 1 कुल पदों की संख्या = 2n + 1

21. Suppose

$$\det\begin{bmatrix}\sum_{k=0}^{n}k & \sum_{k=0}^{n}{}^nC_k k^2\\ \sum_{k=0}^{n}{}^nC_k k & \sum_{k=0}^{n}{}^nC_k 3^k\end{bmatrix}=0$$ **{[BT-BC]-M-305}**

holds for some positive integer n. Then $\sum_{k=0}^{n}\frac{{}^nC_k}{k+1}$ equals.

मानाकि

किसी धनात्मक पूर्णांक n के लिए

$$\det\begin{bmatrix}\sum_{k=0}^{n}k & \sum_{k=0}^{n}{}^nC_k k^2\\ \sum_{k=0}^{n}{}^nC_k k & \sum_{k=0}^{n}{}^nC_k 3^k\end{bmatrix}=0$$, तो $\sum_{k=0}^{n}\frac{{}^nC_k}{k+1}$ का मान है–

[JEE(Advanced) 2019, Paper-2 ,(4, –1)/62]

Ans. **(6.20)** **[Binomial Theorem_M]**

Sol. $\begin{vmatrix} \dfrac{n(n+1)}{2} & n2^{n-1}+n(n-1)2^{n-2} \\ n2^{n-1} & 4^n \end{vmatrix} = 0$

$$\frac{n(n+1)}{2} - \frac{n^2}{4} - \frac{n^2(n-1)}{8} = 0$$

$n = 0$ or $4(n + 1) - 2n - n(n - 1) = 0$

$4n + 4 - 2n - n^2 + n = 0$

$3n - n^2 + 4 = 0 \Rightarrow n^2 - 3n - 4 = 0$

$(n - 4)(n + 1) = 0$

$n = 4$

$$\sum_{r=0}^{4} \frac{{}^4C_r}{r+1} = \sum_{r=0}^{4} \frac{{}^5C_{r+1}}{5} = \frac{2^5-1}{5} = \frac{31}{5} = 6.20$$

PART-II: NUMERICAL VALUE QUESTIONS

भाग-II : संख्यात्मक प्रश्न (NUMERICAL VALUE QUESTIONS)

INSTRUCTION :

- ❖ The answer to each question is **NUMERICAL VALUE** with two digit integer and decimal upto two digit.
- ❖ If the numerical value has more than two decimal places **truncate/round-off** the value to **TWO** decimal placed.

निर्देश :

- ❖ इस खण्ड में प्रत्येक प्रश्न का उत्तर **संख्यात्मक मान** के रूप में है जिसमें दो पूर्णांक अंक तथा दो अंक दशमलव के बाद में है।
- ❖ यदि संख्यात्मक मान में दो से अधिक दशमलव स्थान है, तो संख्यात्मक मान को दशमलव के **दो** स्थानों तक **ट्रंकेट/राउंड ऑफ (truncate/round-off)** करें।

1. If $\dfrac{1}{1!10!} + \dfrac{1}{2!9!} + \dfrac{1}{3!8!} + ... + \dfrac{1}{10!1!} = \dfrac{(2^{10}-1)}{k\ 10!}$ then find the value of k.

यदि $\dfrac{1}{1!10!} + \dfrac{1}{2!9!} + \dfrac{1}{3!8!} + ... + \dfrac{1}{10!1!} = \dfrac{(2^{10}-1)}{k\ 10!}$ तब k का मान ज्ञात कीजिए।

Ans. **05.50**

Sol. $\dfrac{1}{11!}\left[\dfrac{11!}{1!10!} + \dfrac{11!}{2!9!} + \dfrac{11!}{3!10!} + ... + \dfrac{11!}{1!10!}\right] = \dfrac{1}{11!}\ [{}^{11}C_1 + {}^{11}C_2 + ... + {}^{11}C_{10}]$

$= \frac{1}{11!}\,[2^{11}-2] = \frac{2}{11!}\,[2^{10}-1] \Rightarrow k = 11$

2. If the 6th term in the expansion of $\left[\frac{1}{x^{8/3}} + x^2 \log_{10} x\right]^8$ is 5600, then x =

यदि $\left[\frac{1}{x^{8/3}} + x^2 \log_{10} x\right]^8$ के प्रसार में छठवां पद 5600 है, तो x =

Ans. **10.00**

Sol. $T_6 = {}^8C_5\left(\frac{1}{x^{8/3}}\right)^3 (x^2 \log_{10} x)^5 = 5600 \quad \Rightarrow \frac{1}{x^8}\, x^{10} (\log_{10} x)^5 = 100 \quad \Rightarrow x = 10$

3. The number of values of 'x' for which the fourth term in the expansion, $\left(5^{\frac{2}{5}\log_5\sqrt{4^x+44}} + \frac{1}{5^{\log_5\sqrt[3]{2^{x-1}+7}}}\right)^8$ is 336, is :

'x' के मानों की संख्या होगी जिसके लिए व्यंजक $\left(5^{\frac{2}{5}\log_5\sqrt{4^x+44}} + \frac{1}{5^{\log_5\sqrt[3]{2^{x-1}+7}}}\right)^8$ में चौथा पद 336 है –

Ans. **02.00**

Sol. $T_4 = {}^8C_3\left(5^{\frac{1}{5}\log_5(4^x+44)}\right)^5\left(\frac{1}{5^{\frac{1}{3}\log_5(2^{x-1}+7)}}\right)^3 \Rightarrow {}^8C_3(4^x+44)\left(\frac{1}{2^{x-1}+7}\right) = 336$

$\Rightarrow \frac{4^x+44}{2^{x-1}+7} = 6 \quad \Rightarrow \quad 4^x + 44 = 3.2^x + 42 \quad \Rightarrow \quad (2^x)^2 - 3.2^x + 2 = 0$

$\Rightarrow (2^x - 1)(2^x - 2) = 0 \quad \Rightarrow \quad x = 0 \ \& \ 1$

4. If second, third and fourth terms in the expansion of $(x + a)^n$ are 240, 720 and 1080 respectively, then ratio of last term and first term is.

यदि $(x + a)^n$ के विस्तार में दुसरा, तीसरा और चौथा पद क्रमशः 240, 720 तथा 1080 है तब अन्तिम पद तथा प्रथम पद का अनुपात होगा –

Ans. **07.59**

Sol. $T_2 = {}^nC_1\,(x)^{n-1}\,.\,a = 240$ ....(i)

$T_3 = {}^nC_2\,(x)^{n-2}\,a^2 = 720$ ....(ii)

$T_4 = {}^nC_3\,(x)^{n-3}\,a^3 = 1080$ ....(iii)

From (i) and (ii) (i) तथा (ii) से

Here यहाँ $\frac{{}^nC_1(x)^{n-1}a}{{}^nC_2x^{n-2}a^2} = \frac{2x}{(n-1)a} = \frac{240}{720} = \frac{1}{3} \Rightarrow 6x = (n-1)a$

From (ii) and (iii) (ii) तथा (iii) से

$9x = 2(n - 2)\,a$

On dividing भाग देने पर $\frac{3}{2} = \frac{2(n-2)}{(n-1)} \Rightarrow 3n - 3 = 4n - 8 \Rightarrow n = 5$

hence $\frac{T_6}{T_1} = \frac{{}^5C_5(x)^0(a)^5}{{}^5C_0x^5a^0} = \left(\frac{a}{x}\right)^5 = \left(\frac{3}{2}\right)^5 = \mathbf{07.59}$

5. Let the co-efficients of x^n in $(1 + x)^{2n}$ & $(1 + x)^{2n-1}$ be P & Q respectively, then $\left(\frac{P+Q}{P}\right)^4 =$

मानाकि $(1 + x)^{2n}$ एवं $(1 + x)^{2n-1}$ के प्रसार में x^n के गुणांक क्रमशः P एवं Q है, तो $\left(\frac{P+Q}{P}\right)^4 =$

Ans. **05.06**

Sol. $P = {}^{2n}C_n$ and तथा $Q = {}^{2n-1}C_n \Rightarrow \frac{P}{Q} = 2 \ ; \ \left(1+\frac{Q}{P}\right)^4 = (1 + \frac{1}{2})^4 = \frac{81}{16}$

6. In the expansion of $\left(3^{\frac{-x}{4}}+3^{\frac{5x}{4}}\right)^n$, the sum of the binomial coefficients is 256 and four times the term with greatest binomial coefficient exceeds the square of the third term by 21n, then find x.

$\left(3^{\frac{-x}{4}}+3^{\frac{5x}{4}}\right)^n$ के विस्तार में द्विपद गुणाकों का योग 256 है और अधिकतम द्विपद गुणांक का चार गुना, तीसरे पद के वर्ग से 21n, अधिक है तब x का मान ज्ञात कीजिए।

Ans. **00.50**

Sol. $2^n = 256 = 2^8$

$n = 8$

$\left(3^{\frac{-x}{4}}+3^{\frac{5x}{4}}\right)^8$

$4\,T_5 = T_3^2 + 21n$

$4 \times {}^8C_4 \times 3^{\frac{-x}{4}\times 4} \times 3^{\frac{5x}{4}\times 4} = \left({}^8C_2 \times 3^{\frac{-x}{4}\times 6} \times 3^{\frac{5x}{4}\times 2}\right)^2 + 21n$

$1120 \times 3^{x/4} = (28 \times 3^x)^2 + 21n$

$1120 \times 3^{x/4} = 28^2 \times 3^{2x} + 21 \times 8$

$\Rightarrow \quad x = \frac{1}{2}$

7. If $\sum_{k=1}^{19} \frac{(-2)^k}{k!(19-k)!} = \frac{1}{k\ 18!}$ then find k.

यदि $\sum_{k=1}^{19} \frac{(-2)^k}{k!(19-k)!} = \frac{1}{k\ 18!}$ हो तो k का मान ज्ञात कीजिए।

Ans. **09.50**

Sol. $\frac{1}{19!}\sum_{k=1}^{19}(-2)^k.{}^{19}C_k$

$= \frac{1}{19!}\sum_{k=1}^{19}(-1)^k.2^k.{}^{19}C_k$

$= \frac{1}{19!}\left[-{}^{19}C_1.2 + {}^{19}C_2.2^2 - {}^{19}C_3.2^3 + \ldots - 2^{19}.{}^{19}C_{19}\right]$

$= \frac{1}{19!}\left[{}^{19}C_0 - {}^{19}C_1 + {}^{19}C_2.2^2 - \ldots 2^{19}.{}^{19}C_{19} - 1\right]$

$= \frac{1}{19!}\left((1-2)^{19} - 1\right) = \frac{-2}{19!}$

8. The value of p, for which coefficient of x^{50} in the expression $(1 + x)^{1000} + 2x\,(1 + x)^{999} + 3x^2\,(1 + x)^{998} + \ldots.. + 1001\,x^{1000}$ is equal to ${}^{1002}C_p$, is :

व्यंजक $(1 + x)^{1000} + 2x\,(1 + x)^{999} + 3x^2\,(1 + x)^{998} + \ldots.. + 1001\,x^{1000}$ में x^{50} का गुणांक ${}^{1002}C_p$ है, तो p का मान है

Ans. **50.00**

Sol. Co-efficient of x^{50} (x^{50} का गुणांक)

$S = (1 + x)^{1000} + 2x(1 + x)^{999} + 3x^2(1 + x)^{998} + \ldots.. + 1001x^{1000}$ ....(i)

$$\frac{xS}{1+x} = x(1+x)^{999} + 2x^2(1+x)^{998} .... + 1000x^{1000} + \frac{1001 \quad x^{1001}}{(1+x)} \qquad ....(ii)$$

$$\frac{S}{1+x} = (1+x)^{1000} + x(1+x)^{999} + ... + x^{1000} - \frac{1001 \quad x^{1001}}{1+x}$$

$$\Rightarrow \qquad \frac{S}{1+x} = (1+x)^{1000}\left[\frac{1-\left(\frac{x}{1+x}\right)^{1001}}{1-\frac{x}{1+x}}\right] - \frac{1001 \quad x^{1001}}{(1+x)}$$

$$\Rightarrow \qquad S = (1+x)^{1002} - x^{1001}(1+x) - 1001\ x^{1001}$$

Co-efficient of $x^{50} = {}^{1002}c_{50}$ (x^{50} का गुणांक $= {}^{1002}c_{50}$)

9. If $\{x\}$ denotes the fractional part of 'x', and $\left\{\frac{3^{1001}}{82}\right\} = \frac{1}{\lambda}$ then value of λ is

यदि $\{x\}$, 'x' के भिन्नात्मक भाग को प्रदर्शित करता है, तथा $\left\{\frac{3^{1001}}{82}\right\} = \frac{1}{\lambda}$ हो तो λ का मान होगा –

Ans. **27.33**

Sol. $\left\{\frac{3^{1001}}{82}\right\} = \left\{\frac{3.(82-1)^{250}}{82}\right\} = \left\{\frac{3.[{}^{250}C_0(82)^{250} + {}^{250}C_1(82)^{249}(-1) + ... + {}^{250}C_{250}]}{82}\right\} = \frac{3}{82}$

10. The index 'n' of the binomial $\left(\frac{x}{5} + \frac{2}{5}\right)^n$ if the only 9^{th} term of the expansion has numerically the greatest coefficient ($n \in N$) then find $\frac{T_9}{T_8}$ (where T_r denote coefficient of r^{th} term from beginning in the expansion)

यदि $\left(\frac{x}{5} + \frac{2}{5}\right)^n$, ($n \in N$) के प्रसार में केवल 9वाँ पद संख्यात्मक रूप से महत्तम गुणांक वाला पद हो, तो $\frac{T_9}{T_8}$ का मान ज्ञात कीजिए (जहाँ T_r प्रसार में प्रारम्भ से rवें पद के गुणांक को प्रदर्शित करता है)

Ans. **01.25**

Sol. For T_9 to be the numerically greatest term, $r = \left[\frac{n+1}{1+\left|\frac{x}{a}\right|}\right] = \left[\frac{n+1}{1+\left|\frac{1}{2}\right|}\right] = 8$

$$\Rightarrow \qquad 8 < \frac{2(n+1)}{3} < 9 \Rightarrow 11 < n < 12.5 \Rightarrow n = 12$$

then $\frac{T_9}{T_8} = \frac{{}^{12}C_8\left(\frac{1}{5}\right)^4\left(\frac{2}{5}\right)^8}{{}^{12}C_7\left(\frac{1}{5}\right)^5\left(\frac{2}{5}\right)^7} = \frac{5}{4}$

Hindi. चूँकि T_9 संख्यात्मक रूप से महत्तम पद है अतः $r = \left[\frac{n+1}{1+\left|\frac{x}{a}\right|}\right] = \left[\frac{n+1}{1+\left|\frac{1}{2}\right|}\right] = 8 \qquad \Rightarrow \qquad 8 < \frac{2(n+1)}{3} < 9$

$\Rightarrow 11 < n < 12.5 \Rightarrow n = 12$

तब $\frac{T_9}{T_8} = \frac{{}^{12}C_8\left(\frac{1}{5}\right)^4\left(\frac{2}{5}\right)^8}{{}^{12}C_7\left(\frac{1}{5}\right)^5\left(\frac{2}{5}\right)^7} = \frac{5}{4}$

11. Sum of square of all possible values of 'r' satisfying the equation,

$${}^{39}C_{3r-1} - {}^{39}C_{r^2} = {}^{39}C_{r^2-1} - {}^{39}C_{3r} \text{ is :}$$

समीकरण ${}^{39}C_{3r-1} - {}^{39}C_{r^2} = {}^{39}C_{r^2-1} - {}^{39}C_{3r}$ को संतुष्ट करने वाले 'r' के सभी संभावित मानों के वर्गो का योग होगा –

Ans. **34.00**

Sol. ${}^{39}C_{3r-1} - {}^{39}C_{r^2} = {}^{39}C_{r^2-1} - {}^{39}C_{3r}$

$\Rightarrow {}^{39}C_{3r-1} + {}^{39}C_{3r} = {}^{39}C_{r^2-1} + {}^{39}C_{r^2}$

$\Rightarrow {}^{40}C_{3r} = {}^{40}C_{r^2} \Rightarrow$ (i) $r^2 = 3r \Rightarrow r = 0, 3$

(ii) $r^2 + 3r = 40 \Rightarrow r = 5, -8$

possible values are 3,5

12. Find the value of

${}^6C_0 . {}^{12}C_6 - {}^6C_1 {}^{11}C_6 + {}^6C_2 {}^{10}C_6 - {}^6C_3 . {}^9C_6 + {}^6C_4 . {}^8C_6 - {}^6C_5 . {}^7C_6 + {}^6C_6 . {}^6C_6$ का मान ज्ञात कीजिए

Ans. **01.00**

Sol. Coeff of x^6 : ${}^6C_0.(1+x)^{12} - {}^6C_1(1+x)^{11} + {}^6C_2(1+x)^{10} - {}^6C_3(1+x)^9$

$+ {}^6C_4(1+x)^8 - {}^6C_5(1+x)^7 + {}^6C_6(1+x)^6$

$$= (1+x)^{12}\left[{}^6C_0 - {}^6C_1\left(\frac{1}{1+x}\right) + {}^6C_2\left(\frac{1}{1+x}\right)^2 - {}^6C_3\left(\frac{1}{1+x}\right)^3 + {}^6C_4\left(\frac{1}{1+x}\right)^4 - {}^6C_5\left(\frac{1}{1+x}\right)^5 + {}^6C_6\left(\frac{1}{1+x}\right)^6\right]$$

$= (1+x)^{12}.(1 - \frac{1}{1+x})^6$

$= (1+x)^6 . x^6$

$= 1 \times$ coeff of $x^6 = 1$

13. If n is a positive integer & $C_k = {}^nC_k$, find the value of $\left(\sum_{k=1}^{n}\frac{k^3}{n(n+1)^2.(n+2)}\left(\frac{C_k}{C_{k-1}}\right)^2\right)$ is :

यदि n एक धनात्मक पूर्णांक है तथा $C_k = {}^nC_k$ तब $\left(\sum_{k=1}^{n}\frac{k^3}{n(n+1)^2.(n+2)}\left(\frac{C_k}{C_{k-1}}\right)^2\right)$ का मान है–

Ans. **00.08**

Sol $\sum_{k=1}^{n} k^3\left(\frac{n-k+1}{k}\right)^2 = \sum_{k=1}^{n} k(n-k+1)^2 = \sum_{k=1}^{n}(n^2k + k^3 + k - 2nk^2 + 2nk - 2k^2)$

$= \frac{(n+1)^2.n(n+1)}{2} + \left[\frac{n(n+1)}{2}\right]^2 - \frac{2(n+1)n(n+1)(2n+1)}{6} = \frac{n(n+1)^2(n+2)}{12}$

14. The value of the expression $\left(\sum_{r=0}^{10} {}^{10}C_r\right)\left(\sum_{K=0}^{10}(-1)^K \frac{{}^{10}C_K}{2^K}\right)$ is :

व्यंजक $\left(\sum_{r=0}^{10} {}^{10}C_r\right)\left(\sum_{K=0}^{10}(-1)^K \frac{{}^{10}C_K}{2^K}\right)$ का मान है–

Ans. **01.00**

Sol. $\left(\sum_{r=0}^{10} {}^{10}C_r\right)\left(\sum_{k=0}^{10}(-1)^k \frac{{}^{10}C_k}{2^k}\right) = ({}^{10}C_0 + ... + {}^{10}C_{10})\left({}^{10}C_0 - \frac{{}^{10}C_1}{2} + \frac{{}^{10}C_2}{2^2} .... + \frac{{}^{10}C_{10}}{2^{10}}\right) = 2^{10} \times \left(1 - \frac{1}{2}\right)^{10} = 1$

15. The value of λ if $\sum_{m=97}^{100} {}^{100}C_m \,.\, {}^{m}C_{97} = \lambda\, {}^{99}C_{96}$ is :

λ का मान होगा यदि $\sum_{m=97}^{100} {}^{100}C_m \,.\, {}^{m}C_{97} = \lambda\, {}^{99}C_{96}$ है।

Ans. **08.24 or 08.25**

Sol. $\sum_{m=p}^{n} {}^{n}C_m . {}^{m}C_p = \sum_{m=p}^{n} \frac{n\;!}{m\;!(n-m)\;!} \times \frac{m\;!}{p\;!(m-p)\;!} = \sum_{m=p}^{n} {}^{n}C_p . {}^{n-p}C_{m-p}$

$= {}^{n}C_p\,[{}^{n-p}C_0 + {}^{n-p}C_1 + ... + {}^{n-p}C_{n-p}] = {}^{n}C_p\, 2^{n-p}$; where n = 100 and p = 97.

जहाँ n = 100 तथा p = 97

16. If $(1 + x + x^2 + ... + x^p)^n = a_0 + a_1x + a_2x^2 + ... + a_{np}\, x^{np}$, then the value of :

$\frac{1}{p(p+1)^7}\,[a_1 + 2a_2 + 3a_3 + ... + 7p\, a_{7p}]$ is :

यदि $(1 + x + x^2 + ... + x^p)^n = a_0 + a_1x + a_2x^2 + ... + a_{np}\, x^{np}$, हो, तो $\frac{1}{p(p+1)^7}\,[a_1 + 2a_2 + 3a_3 + ... + 7p\, a_{7p}]$ का मान है–

Ans. **03.50**

Sol $(1 + x + x^2 + ... + x^p)^n = a_0 + a_1x + ... + a_{np}\, x^{np}$

Differentiating both side w.r.t. x   x के सापेक्ष अवकलन करने पर

$n(1 + x + x^2 + ... + x^p)^{n-1}(1 + 2x + ... + px^{p-1}) = a_1 + 2a_2x + ... + np\, a_{np}x^{np-1}$

Now put x = 1   रखने पर

$a_1 + 2a_2 + ... + np\, a_{np} = n(p + 1)^{n-1}(1 + 2 + ... + p) = \frac{n(p+1)^n.\; p}{2}$, where $n = \frac{7}{2}$

17. If $({}^{2n}C_1)^2 + 2.\,({}^{2n}C_2)^2 + 3.\,({}^{2n}C_3)^2 + ... + 2n.\,({}^{2n}C_{2n})^2 = 18\, .\, {}^{4n-1}C_{2n-1}$, then n is :

यदि $({}^{2n}C_1)^2 + 2.\,({}^{2n}C_2)^2 + 3.\,({}^{2n}C_3)^2 + ... + 2n.\,({}^{2n}C_{2n})^2 = 18\, .\, {}^{4n-1}C_{2n-1}$, तब n है –

Ans. **09.00**

Sol. $\because$ $(1+x)^{2n} = {}^{2n}C_0 + {}^{2n}C_1x + ... + {}^{2n}C_{2n}x^{2n}$

differentiating it

$2n(1+x)^{2n-1} = {}^{2n}C_1 + 2.\,{}^{2n}C_2x + ... + 2n\,{}^{2n}C_{2n}x^{2n-1}$

Again $(x+1)^{2n} = {}^{2n}C_0x^{2n} + {}^{2n}C_1x^{2n-1} + {}^{2n}C_2x^{2n-2} + ... + {}^{2n}C_{2n}$

Required expression = coefficient of x^{2n-1} in $2n\,(1+x)^{4n-1}$

$= 2n.\, {}^{4n-1}C_{2n-1}$

Hindi $\because$ $(1+x)^{2n} = {}^{2n}C_0 + {}^{2n}C_1x + ... + {}^{2n}C_{2n}x^{2n}$

अवकलन करने पर

$2n(1+x)^{2n-1} = {}^{2n}C_1 + 2.\,{}^{2n}C_2x + \ldots\ldots\ldots\ldots + 2n^{2n}C_{2n}x^{2n-1}$

पुनः $(x+1)^{2n} = {}^{2n}C_0x^{2n} + {}^{2n}C_1x^{2n-1} + {}^{2n}C_2x^{2n-2} + \ldots\ldots\ldots\ldots + {}^{2n}C_{2n}$

अभीष्ट व्यंजक $= 2n\,(1+x)^{4n-1}$ में x^{2n-1} का गुणांक

$= 2n.\,{}^{4n-1}C_{2n-1}$

18. If $\sum\limits_{r=0}^{n} \frac{2r+3}{r+1}.{}^nC_r = \frac{(2n+3k)2^n - 1}{n+1}$ then 'k' is

यदि $\sum\limits_{r=0}^{n} \frac{2r+3}{r+1}.{}^nC_r = \frac{(2n+3k)2^n - 1}{n+1}$ है तब 'k' का मान है

Ans. **01.33**

Sol. $\sum\limits_{r=0}^{n} \frac{2r+3}{r+1}.{}^nC_r = \sum\limits_{r=0}^{n} 2.{}^nC_r + \sum\limits_{r=0}^{n} \frac{1}{r+1}.{}^nC_r = 2\,.2^2 + \frac{1}{n+1}\,.\,\sum\limits_{r=0}^{n}.{}^{n+1}C_{r+1}$

$= 2^{n+1} + \frac{1}{n+1}.(2^{n+1} - 1) = \frac{(n+2).2^{n+1} - 1}{n+1}$

19. If $\sum\limits_{r=0}^{n} \frac{(-1)^r.C_r}{(r+1)(r+2)(r+3)} = \frac{a}{(n+b)}$, then a + b is

यदि $\sum\limits_{r=0}^{n} \frac{(-1)^r.C_r}{(r+1)(r+2)(r+3)} = \frac{a}{(n+b)}$ है, तब a + b का मान है

Ans. **03.50**

Sol. $\sum\limits_{r=0}^{n} \frac{(-1)^r.C_r}{(r+1)(r+2)(r+3)} = \frac{-1}{(n+1)(n+2)(n+3)}$

$\frac{-1}{(n+1)(n+2)(n+3)}\left[(1-1)^{n+3} - \left\{{}^{n+3}C_0(-1)^0 + {}^{n+3}C_1(-1)^1 + {}^{n+3}C_2(-1)^2\right\}\right]$

$\frac{1}{(n+1)(n+2)(n+3)}\ (n+2) \times \frac{(n+1)}{2} = \frac{1}{2(n+3)}$

20. $\sum\limits_{k=1}^{3n} {}^{6n}C_{2k-1}\,(-3)^k$ is equal to :

$\sum\limits_{k=1}^{3n} {}^{6n}C_{2k-1}\,(-3)^k$ बराबर है

Ans. **00.00**

Sol. $S = \sum\limits_{k=1}^{3n} {}^{6n}C_{2k-1}(-3)^k \quad \Rightarrow \quad S = {}^{6n}C_1(-3) + {}^{6n}C_3\,(-3)^2 + \ldots\ldots. + {}^{6n}C_{6n-1}(-3)^{3n}$

$\Rightarrow S = (\sqrt{3}\ \ i)\sum\limits_{k=1}^{3n} {}^{6n}C_{2k-1}(\sqrt{3}\ \ i)^{2k-1} \qquad \Rightarrow S = (\sqrt{3}\ \ i)\left[\frac{(1+\sqrt{3}\ \ i)^{6n} - (1-\sqrt{3}\ \ i)^{6n}}{2}\right] = 0$

21. If x is very large as compare to y, then the value of k in $\sqrt{\frac{x}{x+y}}\sqrt{\frac{x}{x-y}} = 1+\frac{ky^2}{x^2}$

यदि x, y की तुलना में बहुत बड़ा हो, तो $\sqrt{\frac{x}{x+y}}\sqrt{\frac{x}{x-y}} = 1+\frac{ky^2}{x^2}$ में k का मान है

Ans. **00.50**

Sol. $\sqrt{\frac{x}{x+y}}\sqrt{\frac{x}{x-y}} = \left(\frac{1}{1+\frac{y}{x}}\right)^{1/2}\left(\frac{1}{1-\frac{y}{x}}\right)^{1/2} = \left(1-\frac{y^2}{x^2}\right)^{-1/2} = 1+\frac{1}{2}\cdot\frac{y^2}{x^2} \Rightarrow k = 2$

PART - III : ONE OR MORE THAN ONE OPTIONS CORRECT TYPE

भाग - III : एक या एक से अधिक सही विकल्प प्रकार

1. In the expansion of $\left(\sqrt[3]{4}+\frac{1}{\sqrt[4]{6}}\right)^{20}$

(A*) the number of irrational terms is 19 (B*) middle term is irrational
(C*) the number of rational terms is 2 (D*) 9th term is rational

$\left(\sqrt[3]{4}+\frac{1}{\sqrt[4]{6}}\right)^{20}$ के प्रसार में

(A) अपरिमेय पदों की संख्या 19 है। (B) मध्य पद अपरिमेय है।
(C) परिमेय पदों की संख्या 2 है। (D) 9वाँ पद परिमेय है।

Sol. $\left(\sqrt[3]{4}+\frac{1}{\sqrt[4]{6}}\right)^{20}$

$T_{r+1} = {}^{20}C_r (4^{1/3})^{20-r}(6^{-1/4})^r$

For rational terms

$20 - r = 3k$ & $r = 4p$, where $k, p \in I$ $\Rightarrow$ $r = 20$ & $r = 8$

$\therefore$ no. of rational terms = 2 $\therefore$ no. of irrational terms = 19

Hindi $\left(\sqrt[3]{4}+\frac{1}{\sqrt[4]{6}}\right)^{20}$

$T_{r+1} = {}^{20}C_r (4^{1/3})^{20-r}(6^{-1/4})^r$

परिमेय पदों हेतु

$20 - r = 3k$ तथा $r = 4p$, जहाँ $k, p \in I$

$\Rightarrow r = 20$ तथा $r = 8$ $\therefore$ परिमेय पदों की संख्या = 2 $\therefore$ अपरिमेय पदों की संख्या = 19

2. The coefficient of x^4 in $\left(\frac{1+x}{1-x}\right)^2$, $|x| < 1$, is

$\left(\frac{1+x}{1-x}\right)^2$, $|x| < 1$ में x^4 का गुणांक है–

(A) 4 (B) –4 (C*) $10 + {}^4C_2$ (D*) 16

Sol. $(1 + x)^2 (1 - x)^{-2} = (1 + x^2 + 2x)(1 - x)^{-2}$

Co-efficient of x^4 का गुणांक $= {}^5C_4 + {}^3C_2 + 2\,{}^4C_3 = 16$

3. $7^9 + 9^7$ is divisible by :

(A*) 16 (B) 24 (C*) 64 (D) 72

$7^9 + 9^7$ विभाजित हैं –

(A) 16 से (B) 24 से (C) 64 से (D) 72 से

Sol. $7^9 + 9^7 = (8-1)^9 + (8+1)^7 = {}^9C_0(8)^9 - {}^9C_1(8)^8 + {}^9C_2(8)^7 .....+ {}^9C_8(8) - {}^9C_9 + {}^7C_0(8)^7 +......+ {}^7C_6(8) + {}^7C_7$
This is divisible by 64 & 16 यह 64 और 16 से भाजित है

4. The sum of the series $\sum_{r=1}^{n}(-1)^{r-1}.\ {}^nC_r(a-r)$ is equal to :

(A*) 5 if a = 5 (B) –5 if a = 5 (C*) –5 if a = –5 (D) 5 if a = –5

श्रेणी $\sum_{r=1}^{n}(-1)^{r-1}.\ {}^nC_r(a-r)$ का योगफल बराबर है

(A*) 5 यदि a = 5 (B) –5 यदि a = 5 (C*) –5 यदि a = –5 (D) 5 यदि a = –5

Sol. $= a\sum_{r=1}^{n}(-1)^{r-1}.\ {}^nC_r - \sum_{r=1}^{n} r.\ {}^nC_r(-1)^{r-1} = a[{}^nC_1 - {}^nC_2 + {}^nC_3 .....+(-1)^{n-1}.{}^nC_n] - n\sum_{r=1}^{n}(-1)^{r-1}\ {}^{n-1}C_{r-1}$
$= a(1) - n[{}^{n-1}C_0 - {}^{n-1}C_1 + ...+(-1)^{(n-1)}\,{}^{n-1}C_{n-1}] = a - n(0) = a$

5. Let $a_n = \frac{1000^n}{n!}$ for $n \in N$, then a_n is greatest, when

मानाकि $n \in N$ के लिए $a_n = \frac{1000^n}{n!}$ हो, तो a_n महत्तम होगा, यदि

(A) n = 997 (B) n = 998 (C*) n = 999 (D*) n = 1000

Sol. $a_n = \frac{(1000)(1000)............(1000)}{1.2.........n}$
$a_{999} = a_{1000}$
a_n is maximum for n = 999 and n = 1000

Hindi $a_n = \frac{(1000)(1000)............(1000)}{1.2.........n}$
$a_{999} = a_{1000}$
a_n , n = 999 तथा n = 1000 के लिए अधिकतम है।

6. ${}^nC_0 - 2.3\ {}^nC_1 + 3.3^2\ {}^nC_2 - 4.3^3\ {}^nC_3 +..........+ (-1)^n (n+1)\ {}^nC_n\ 3^n$ is equal to

(A*) $2^n\left(\frac{3n}{2}+1\right)$ if n is even (B) $2^n\left(n+\frac{3}{2}\right)$ if n is even

(C*) $-\ 2^n\left(\frac{3n}{2}+1\right)$ if n is odd (D) $2^n\left(n+\frac{3}{2}\right)$ if n is odd

${}^nC_0 - 2.3\ {}^nC_1 + 3.3^2\ {}^nC_2 - 4.3^3\ {}^nC_3 +..........+ (-1)^n (n+1)\ {}^nC_n\ 3^n$ का मान बराबर है

(A*) $2^n\left(\frac{3n}{2}+1\right)$ यदि n सम है। (B) $2^n\left(n+\frac{3}{2}\right)$ यदि n सम है।

(C*) $-\ 2^n\left(\frac{3n}{2}+1\right)$ यदि n विषम है। (D) $2^n\left(n+\frac{3}{2}\right)$ यदि n विषम है।

Sol. $(1+x)^n = {}^nC_0 + {}^nC_1 x + {}^nC_2 x^2 +..........+ {}^nC_n x^n$
Multiply it by x
$x(1+x)^n = {}^nC_0 x + {}^nC_1 x^2 + {}^nC_2x^3 +............ {}^nC_nx^{n+1}$
Differentiate w.r. to x and put x = – 3
$n x (1+x)^{n-1} + (1+x)^n = {}^nC_0 + 2{}^nC_1 x + 3\ {}^nC_2 x^2 + 4\ {}^nC_3 x^3 +........+ (n+1)\ {}^nC_n x^n$
So answer, $-3n(-2)^{n-1} + (-2)^n$

$= (-2)^n\left(1+\frac{3n}{2}\right) = (-1)^n 2^n\left(\frac{3n}{2}+1\right)$

Hindi $(1 + x)^n = {}^nC_0 + {}^nC_1 x + {}^nC_2 x^2 + ... + {}^nC_n x^n$

x से गुणा करने पर

$x(1 + x)^n = {}^nC_0 x + {}^nC_1 x^2 + {}^nC_2x^3 + ... {}^nC_nx^{n+1}$

x के सापेक्ष अवकलन करके $x = -3$ रखने पर

$n x (1 + x)^{n-1} + (1 + x)^n = {}^nC_0 + 2{}^nC_1 x + 3\,{}^nC_2 x^2 + 4\,{}^nC_3 x^3 + ... + (n + 1)\,{}^nC_n x^n$

अतः उत्तर $-3n(-2)^{n-1} + (-2)^n = (-2)^n\left(1+\frac{3n}{2}\right) = (-1)^n 2^n\left(\frac{3n}{2}+1\right)$

7. Element in set of values of r for which, ${}^{18}C_{r-2} + 2.\,{}^{18}C_{r-1} + {}^{18}C_r \geq {}^{20}C_{13}$ is :

r के मानों की संख्या होगी जिसके लिए ${}^{18}C_{r-2} + 2.\,{}^{18}C_{r-1} + {}^{18}C_r \geq {}^{20}C_{13}$ है

(A*) 9 (B) 5 (C*) 7 (D*) 10

Sol. ${}^{18}C_{r-2} + 2.{}^{18}C_{r-1} + {}^{18}C_r \geq {}^{20}C_{13}$

or या ${}^{19}C_{r-1} + {}^{19}C_r \geq {}^{20}C_{13}$

or या ${}^{20}C_r \geq {}^{20}C_{13}$

r = 7,8,9,10,11,12,13

8. The expansion of $(3x + 2)^{-1/2}$ is valid in ascending powers of x, if x lies in the interval.

$(3x + 2)^{-1/2}$ का प्रसार x की बढ़ती हुई घातों के लिए वैध है यदि x अन्तराल में स्थित होगा।

(A*) (0, 2/3) (B) (–3/2, 3/2)

(C*) (–2/3, 2/3) (D) (–∞, –3/2) (3/2, ∞)

Sol. $(3x + 2)^{-1/2}$ has infinite expansion when $\left|\frac{3x}{2}\right| < 1 \Rightarrow x \in \left(-\frac{2}{3}, \frac{2}{3}\right)$

Hindi. $(3x + 2)^{-1/2}$ का प्रसार अनन्त पदों तक होगा यदि $\left|\frac{3x}{2}\right| < 1 \Rightarrow x \in \left(-\frac{2}{3}, \frac{2}{3}\right)$

9. If $(1 + 2x + 3x^2)^{10} = a_0 + a_1x + a_2x^2 + ... + a_{20}x^{20}$, then :

यदि $(1 + 2x + 3x^2)^{10} = a_0 + a_1x + a_2x^2 + ... + a_{20}x^{20}$ हो, तो

(A*) $a_1 = 20$ (B*) $a_2 = 210$

(C*) $a_4 = 8085$ (D) $a_{20} = 2^2.\,3^7.\,7$

Sol. General term व्यापक पद $= \frac{10!}{r_1!\ r_2!\ r_3!}(1)^{r_1}(2x)^{r_2}(3x^2)^{r_3}$

a_1 = Coeff. of x (x का गुणांक)

$r_2 + 2r_3 = 1 \Rightarrow r_2 = 1, r_1 = 9, r_3 = 0 \quad \therefore \quad a_1 = \frac{10!}{1!\,9!}(2)^1 = 20$

a_2 = Coeff. of x^2 (x^2 का गुणांक)

$r_2 + 2r_3 = 2 \Rightarrow r_2 = 2, r_1 = 8, r_3 = 0$

$r_2 = 0, r_1 = 9, r_3 = 1$

$a_2\ \frac{10!}{2!\ 8!} = (2)^2 + \frac{10!}{9!\ 1!}(3) = 210$

a_4 = coeff. of x^4 (x^4 का गुणांक)

$r_2 + 2r_3 = 4 \Rightarrow r_2 = 4, r_1 = 6, r_3 = 0$

$r_2 = 2, r_1 = 7, r_3 = 1$

$r_2 = 0, r_1 = 8, r_3 = 2$

$a_4 = \frac{10!}{4!\,6!}\,(2)^4 + \frac{10!}{2!7!1!}\,(2)^2\,(3) + \frac{10!}{8!2!}\,(3)^2 = 8085$

$a_{20} = 3^{10}$

10. In the expansion of $(x + y + z)^{25}$

(A*) every term is of the form ${}^{25}C_r.\,{}^{r}C_k.\,x^{25-r}.\,y^{r-k}.\,z^k$ (B*) the coefficient of $x^8\,y^9\,z^9$ is 0

(C) the number of terms is 325 (D) none of these

$(x + y + z)^{25}$ के प्रसार में

(A) प्रत्येक पद ${}^{25}C_r.\,{}^{r}C_k.\,x^{25-r}.\,y^{r-k}.\,z^k$ रूप में होगा। (B) $x^8\,y^9\,z^9$ का गुणांक 0 है।

(C) पदों की संख्या 325 है। (D) इनमें से कोई नहीं

Sol. $(x + y + z)^{25}$

General term व्यापक पद $= \frac{25!}{r_1!\,r_2!\,r_3!}\,x^{r_1}y^{r_2}z^{r_3}$

Putting $r_3 = k$, $r_2 = r - k$ and तथा $r_1 = 25 - r$ रखने पर

$= \frac{25!}{(25-r)!(r-k)!(k)!} \times \frac{r!}{r!} \times x^{25-r}\,y^{r-k}\,z^k = {}^{25}C_r\,.\,{}^{r}C_k\,.\,x^{25-r}\,y^{r-k}\,z^k$

$r_1 + r_2 + r_3 = 25$

$\therefore$ coefficient of $x^8\,y^9\,z^9$ is 0 $\therefore$ $x^8\,y^9\,z^9$ का गुणांक = 0

11. If $(1 + x + 2x^2)^{20} = a_0 + a_1x + a_2x^2 + \dots\dots + a_{40}x^{40}$, then $a_0 + a_2 + a_4 \dots\dots + a_{38}$ is equal to :

यदि $(1 + x + 2x^2)^{20} = a_0 + a_1x + a_2x^2 + \dots\dots + a_{40}x^{40}$ हो, तो $a_0 + a_2 + a_4 \dots\dots + a_{38}$ बराबर है –

(A) $2^{19}\,(2^{30} + 1)$ (B*) $2^{19}(2^{20} - 1)$ (C*) $2^{39} - 2^{19}$ (D) $2^{39} + 2^{19}$

Sol. $(1 + x + 2x^2)^{20} = a_0 + a_1x + \dots\dots + a_{40}x^{40}$

$x = 1$, then $a_0 + a_1 + \dots\dots + a_{40} = 4^{20}$

$x = -1$, then $a_0 - a_1 + a_2 - \dots\dots + a_{40} = 2^{20}$

$2^{20} + 2^{40} = 2[a_0 + a_2 + \dots\dots + a_{38} + a_{40}]$

$\Rightarrow a_0 + a_2 + \dots\dots + a_{38} = 2^{19} + 2^{39} - 2^{20} = 2^{19}\,(2^{20}-1) \quad \therefore\ a_{40} = a^{20}$

Hindi. $(1 + x + 2x^2)^{20} = a_0 + a_1x + \dots\dots + a_{40}x^{40}$

$x = 1$, तो $a_0 + a_1 + \dots\dots + a_{40} = 4^{20}$

$x = -1$, तो $a_0 - a_1 + a_2 - \dots\dots + a_{40} = 2^{20}$

$2^{20} + 2^{40} = 2[a_0 + a_2 + \dots\dots + a_{38} + a_{40}]$

$\Rightarrow a_0 + a_2 + \dots\dots + a_{38} = 2^{19} + 2^{39} - 2^{20} = 2^{19}\,(2^{20}-1) \quad \therefore\ a_{40} = a^{20}$

12. $n^n\left(\frac{n+1}{2}\right)^{2n}$ is $(n \in N)$

(A) Less than $\left(\frac{n+1}{2}\right)^3$ (B*) Greater than or equal to $\left(\frac{n+1}{2}\right)^3$

(C) Less than $(n!)^3$ (D*) Greater than or equal to $(n!)^3$.

$n^n\left(\frac{n+1}{2}\right)^{2n}$ है– $(n \in N)$

(A) $\left(\frac{n+1}{2}\right)^3$ से छोटा (B*) $\left(\frac{n+1}{2}\right)^3$ से बड़ा या बराबर

(C) $(n!)^3$ से छोटा (D*) $(n!)^3$ से बड़ा या बराबर

Sol. $n^n\left(\frac{n+1}{2}\right)^{2n} = \left(\frac{\left(\frac{n(n+1)}{2}\right)^2}{n}\right)^n = \left(\frac{1^3+2^3+...+n^3}{n}\right)^n$; $\frac{1^3+2^3+....+n^3}{n} \geq \sqrt[n]{(n!)^3}$

13. If recursion polynomials $P_k(x)$ are defined as $P_1(x) = (x-2)^2$, $P_2(x) = ((x-2)^2-2)^2$
$P_3(x) = ((x-2)^2-2)^2-2)^2$ .... (In general $P_k(x) = (P_{k-1}(x)-2)^2$, then the constant term in $P_k(x)$ is

(A*) 4 (B) 2 (C) 16 (D*) a perfect square

यदि व्यंजक $P_k(x)$ इस तरह से परिभाषित है कि $P_1(x) = (x-2)^2$, $P_2(x) = ((x-2)^2-2)^2$
$P_3(x) = ((x-2)^2-2)^2-2)^2$ (व्यापक रूप से $P_k(x) = (P_{k-1}(x)-2)^2$), तो $P_k(x)$ में अचर पद है–

(A*) 4 (B) 2 (C) 16 (D*) एक पूर्ण वर्ग

Sol. Constant term in $P_1(x)$ is 4
If the constant term in $P_k(x)$ is also 4, then
$P_k(x) = 4 + a_1x + a_2x^2 + ...$
and $P_{k+1}(x) = (P_k(x)-2)^2 = (a_1x + a_2x^2 + ...+ 2)^2$

Hindi $P_1(x)$ में नियत पद 4 है।
$P_k(x)$ में भी नियत पद 4 है, तो
$P_k(x) = 4 + a_1x + a_2x^2 + ...$
और $P_{k+1}(x) = (P_k(x)-2)^2 = (a_1x + a_2x^2 + ...+ 2)^2$

PART - IV : COMPREHENSION

भाग - IV : अनुच्छेद (COMPREHENSION)

Comprehension # 1 (Q. No. 1 to 3)

Consider, sum of the series $\sum\sum_{0 \le i < j \le n} f(i)\,f(j)$

In the given summation, i and j are not independent.

In the sum of series $\sum_{i=1}^{n}\sum_{j=1}^{n} f(i)\;f(j) = \sum_{i=1}^{n}\left(f(i)\left(\sum_{j=1}^{n} f(j)\right)\right)$ i and j are independent. In this summation, three types of terms occur, those when i < j, i > j and i = j.

Also, sum of terms when i < j is equal to the sum of the terms when i > j if f(i) and f(j) are symmetrical. So, in that case

$$\sum_{i=1}^{n}\sum_{j=1}^{n} f(i)f(j) = \sum\sum_{0\le i<j\le n} f(i)f(j) + \sum\sum_{0\le i<j\le n} f(i)f(j) + \sum\sum_{i=j} f(i)f(j)$$

$$= 2\sum\sum_{0\le i<j\le n} f(i)f(j) + \sum\sum_{i=j} f(i)f(j)$$

$$\Rightarrow \sum\sum_{0\le i<j\le n} f(i)f(j) = \frac{\sum_{i=0}^{n}\sum_{j=0}^{n} f(i)f(j) - \sum\sum_{i=j} f(i)f(j)}{2}$$

When f(i) and f(j) are not symmetrical, we find the sum by listing all the terms.

अनुच्छेद # 1 (Q. No. 1 to 3)

माना कि श्रेणियों का योगफल $\sum\sum_{0 \le i < j \le n} f(i)\, f(j)$ सूत्र से दिया जाता है जहाँ i तथा j स्वतंत्र नहीं है।

श्रेणियों के योगफल $\sum_{i=1}^{n}\sum_{j=1}^{n} f(i)\quad f(j) = \sum_{i=1}^{n}\left(f(i)\left(\sum_{j=1}^{n} f(j)\right)\right)$ में i व j स्वतंत्र है। इस योगफल में तीन प्रकार के पद होते है। जिनमें i < j, i > j तथा i = j तथा जब i < j के लिए पदो का योगफल, i > j के लिए पदों के योगफल के बराबर है। यदि f(i) तथा f(j) सममित है। इस स्थिति में

$$\sum_{i=1}^{n}\sum_{j=1}^{n} f(i)f(j) = \sum\sum_{0\le i<j\le n} f(i)f(j)$$

$$+ \sum\sum_{0\le i<j\le n} f(i)f(j) + \sum\sum_{i=j} f(i)f(j)$$

$$= 2\sum\sum_{0\le i<j\le n} f(i)f(j) + \sum\sum_{i=j} f(i)f(j)$$

$$\Rightarrow \qquad \sum\sum_{0\le i<j\le n} f(i)f(j) = \frac{\sum_{i=0}^{n}\sum_{j=0}^{n} f(i)f(j) - \sum\sum_{i=j} f(i)f(j)}{2}$$

जब f(i) तथा f(j) सममित नहीं है। तब हम सभी पदों का योगफल ज्ञात करते है।

1. $\sum\sum_{0\le i<j\le n} {}^nC_i \;\; {}^nC_j$ is equal to

$\sum\sum_{0\le i<j\le n} {}^nC_i \;\; {}^nC_j$ का मान बराबर है–

(A*) $\frac{2^{2n} - {}^{2n}C_n}{2}$ (B) $\frac{2^{2n} + {}^{2n}C_n}{2}$ (C) $\frac{2^{2n} - {}^{n}C_n}{2}$ (D) $\frac{2^{2n} + {}^{n}C_n}{2}$

Sol. $\sum\sum_{0\le i<j\le n} {}^nC_i \;\; {}^nC_j$

$$= \frac{\left(\sum_{i=0}^{n}\sum_{j=0}^{n} {}^nC_i \;\; {}^nC_j\right) - \sum_{i=0}^{n} ({}^nC_i)^2}{2} = \frac{\left(\sum_{i=0}^{n} {}^nC_i \;\; 2^n\right) - \sum_{i=0}^{n} ({}^nC_i)^2}{2} = \frac{2^n 2^n - \sum_{i=0}^{n} ({}^nC_i)^2}{2} = \frac{2^{2n} - {}^{2n}C_n}{2}$$

2. Let ${}^0C_0 = 1$, then $\sum_{m=0}^{n}\;\sum_{p=0}^{m} {}^nC_m \;.\; {}^mC_p$ is equal to

माना ${}^0C_0 = 1$, तब $\sum_{m=0}^{n}\;\sum_{p=0}^{m} {}^nC_m \;.\; {}^mC_p$ का मान बराबर है–

(A) $2^n - 1$ (B*) 3^n (C) $3^n - 1$ (D) 2^n

Sol. $\sum_{m=0}^{n}\;\sum_{p=0}^{m} {}^nC_m \;.\; {}^mC_p = \sum_{m=0}^{n} {}^nC_m \left(\sum_{p=0}^{m} {}^mC_p\right) = \sum_{m=0}^{n} {}^nC_m \left(2^m\right) = 3^n$

3. $\sum\sum_{0\le i\le j\le n}\left({}^nC_i + {}^nC_j\right)$

(A*) $(n + 2)2^n$ (B) $(n + 1)2^n$ (C) $(n - 1)2^n$ (D) $(n + 1)2^n - 1$

Sol. $\sum\sum_{0\le i\le j\le n}\left({}^nC_i + {}^nC_j\right) = \dfrac{\left(\sum_{i=0}^{n}\sum_{j=0}^{n}\left({}^nC_i + {}^nC_j\right)\right) - \sum_{i=0}^{n} 2\ {}^nC_i}{2} = \dfrac{\left(\sum_{i=0}^{n}\left(\sum_{j=0}^{n}{}^nC_i + \sum_{j=0}^{n}{}^nC_j\right)\right) - 2\times 2^n}{2}$

$$= \frac{\left(\sum_{i=0}^{n}\left({}^nC_i \sum_{j=0}^{n} 1 + 2^n\right)\right) - 2^{n+1}}{2} = \frac{\left(\sum_{i=0}^{n}\left({}^nC_i(n+1) + 2^n\right)\right) - 2^{n+1}}{2}$$

$$= \frac{(n+1)\sum_{i=1}^{n} {}^nC_i + 2^n\sum_{i=0}^{n} 1 - 2^{n+1}}{2} = \frac{(n+1)2^n + 2^n(n+1) - 2^{n+1}}{2}$$

$= (n + 1)2^n - 2^n = n2^n$

Comprehension # 2 (Q. No. 4 to 6)
अनुच्छेद # 2 (Q. No. 4 to 6) **[Revision Planner]**

Let P be a product given by $P = (x + a_1)(x + a_2) \ldots\ldots\ldots (x + a_n)$

and Let $S_1 = a_1 + a_2 + \ldots\ldots. + a_n = \sum_{i=1}^{n} a_i$, $S_2 = \sum\sum_{i<j} a_i.a_j$, $S_3 = \sum\sum_{i<j<k}\sum a_i.a_j.a_k$ and so on,

then it can be shown that

$P = x^n + S_1 x^{n-1} + S_2 x^{n-2} + \ldots\ldots\ldots + S_n$.

यदि एक गुणनफल P इस प्रकार है, $P = (x + a_1)(x + a_2) \ldots\ldots\ldots (x + a_n)$

तथा माना ∘ $S_1 = a_1 + a_2 + \ldots\ldots. + a_n = \sum_{i=1}^{n} a_i$, $S_2 = \sum\sum_{i<j} a_i.a_j$, $S_3 = \sum\sum_{i<j<k}\sum a_i.a_j.a_k$ इसी प्रकार आगे,

तो

निम्न को सिद्ध किया जा सकता है –

$P = x^n + S_1 x^{n-1} + S_2 x^{n-2} + \ldots\ldots\ldots + S_n$.

4. The coefficient of x^8 in the expression $(2 + x)^2 (3 + x)^3 (4 + x)^4$ must be
व्यंजक $(2 + x)^2 (3 + x)^3 (4 + x)^4$ में x^8 का गुणांक है–
(A) 26 (B) 27 (C) 28 (D*) 29

Sol. The expression $(2 + x)^2 (3 + x)^3 (4 + x)^4 = (x + 2)(x + 2)(x + 3)(x + 3)(x + 3)(x + 4)(x + 4)(x + 4)(x + 4)$
$= x^9 + (2 + 2 + 3 + 3 + 3 + 4 + 4 + 4 + 4)\, x^8 + \ldots\ldots\ldots\ldots$
$\Rightarrow$ Co-efficient of $x^8 = 29$

Hindi. व्यंजक $(2 + x)^2 (3 + x)^3 (4 + x)^4 = (x + 2)(x + 2)(x + 3)(x + 3)(x + 3)(x + 4)(x + 4)(x + 4)(x + 4)$
$= x^9 + (2 + 2 + 3 + 3 + 3 + 4 + 4 + 4 + 4)\, x^8 + \ldots\ldots\ldots\ldots$
$\Rightarrow$ x^8 का गुणांक $= 29$

5. The coefficient of x^{203} in the expression $(x - 1)(x^2 - 2)(x^3 - 3) \ldots\ldots\ldots. (x^{20} - 20)$ must be
व्यंजक $(x - 1)(x^2 - 2)(x^3 - 3) \ldots\ldots\ldots. (x^{20} - 20)$ में x^{203} का गुणांक है–
(A) 11 (B) 12 (C*) 13 (D) 15

Sol. Expression $= x.\, x^2.\, x^3 \ldots\ldots\ldots. x^{20} \left(1-\frac{1}{x}\right)\left(1-\frac{2}{x^2}\right)\left(1-\frac{3}{x^3}\right)\ldots\ldots.\left(1-\frac{20}{x^{20}}\right)$

Let $E = \left(1-\frac{1}{x}\right)\left(1-\frac{2}{x^2}\right)\left(1-\frac{3}{x^3}\right)\ldots\ldots.\left(1-\frac{20}{x^{20}}\right)$

Now Co-efficient of x^{203} in original expression

$\Rightarrow$ Co-efficient of x^{-7} in E.
But

$$E = 1 - \left(\frac{1}{x}+\frac{2}{x^2}+\frac{3}{x^3}+......\right) + \left(\frac{1}{x}.\frac{6}{x^6}+\frac{2}{x^2}.\frac{5}{x^5}+\frac{3}{x^3}.\frac{4}{x^4}+......\right) - \left(\frac{1}{x}.\frac{2}{x^2}.\frac{4}{x^4}+......\right)$$

= Co-efficient of x^{-7} = –7 + 6 + 10 + 12 – 8 = 13

Hindi. व्यंजक = $x.\ x^2.\ x^3\ ....x^{20}\left(1-\frac{1}{x}\right)\left(1-\frac{2}{x^2}\right)\left(1-\frac{3}{x^3}\right).......\left(1-\frac{20}{x^{20}}\right)$

माना $E = \left(1-\frac{1}{x}\right)\left(1-\frac{2}{x^2}\right)\left(1-\frac{3}{x^3}\right).......\left(1-\frac{20}{x^{20}}\right)$

अब मूल व्यंजक में x^{203} का गुणांक = E में x^{-7} का गुणांक
परन्तु

$$E = 1 - \left(\frac{1}{x}+\frac{2}{x^2}+\frac{3}{x^3}+......\right) + \left(\frac{1}{x}.\frac{6}{x^6}+\frac{2}{x^2}.\frac{5}{x^5}+\frac{3}{x^3}.\frac{4}{x^4}+......\right) - \left(\frac{1}{x}.\frac{2}{x^2}.\frac{4}{x^4}+......\right)$$

$\Rightarrow$ x^{-7} का गुणांक = –7 + 6 + 10 + 12 – 8 = 13

6. The coefficient of x^{98} in the expression of (x – 1) (x – 2) .... (x – 100) must be
(A) $1^2 + 2^2 + 3^2 + ... + 100^2$
(B) $(1 + 2 + 3 + ... + 100)^2 - (1^2 + 2^2 + 3^2 + ... + 100^2)$
(C*) $\frac{1}{2}\ [(1 + 2 + 3 + ... + 100)^2 - (1^2 + 2^2 + 3^2 + ... + 100^2)]$
(D) None of these
व्यंजक (x – 1) (x – 2) .... (x – 100) में x^{98} का गुणांक है–
(A) $1^2 + 2^2 + 3^2 + ... + 100^2$
(B) $(1 + 2 + 3 + ... + 100)^2 - (1^2 + 2^2 + 3^2 + ... + 100^2)$
(C*) $\frac{1}{2}[(1 + 2 + 3 + ... + 100)^2 - (1^2 + 2^2 + 3^2 + ... + 100^2)]$
(D) इनमें से कोई नहीं

Sol. The Co-efficient of x^{98} = (1.2 + 2.3 + ...99.100)
= Sum of product of first 100 natural numbers taken two at a time
$= \frac{1}{2}\ [(1 + 2 + 3 + ...+ 100^2)^2 - (1^2 + 2^2 + 3^2 +..........+ 100^2)]$

Hindi. x^{98} का गुणांक = (1.2 + 2.3 + ...99.100)
= दो –दो एक साथ लेने पर प्रथम 100 प्राकृत संख्याओं के गुणनफलों का योगफल
$= \frac{1}{2}\ [(1 + 2 + 3 + ...+ 100^2)^2 - (1^2 + 2^2 + 3^2 +..........+ 100^2)]$

Comprehension # 3 (Q.No. 7 to 9)

Let $(7 + 4\sqrt{3})^n = I + f = {}^nC_0.7^n + {}^nC_1.7^{n-1}.(4\sqrt{3})^1 + ...$ ....(i)
where I & f are its integral and fractional parts respectively.
It means $0 < f < 1$
Now, $0 < 7 - 4\sqrt{3} < 1$ $\Rightarrow$ $0 < (7 - 4\sqrt{3})^n < 1$
Let $(7 - 4\sqrt{3})^n = f' = {}^nC_0.7^n - {}^nC_1.7^{n-1}.(4\sqrt{3})^1 + ...$ ....(ii)
$\Rightarrow$ $0 < f' < 1$
Adding (i) and (ii) (so that irrational terms cancelled out)
$I + f + f' = (7 + 4\sqrt{3})^n + (7 - 4\sqrt{3})^n$
$= 2\ [{}^nC_0\ 7^n + {}^nC_2\ 7^{n-2}\ (4\sqrt{3})^2 + ...]$
$I + f + f'$ = even integer $\Rightarrow$ (f + f′ must be an integer)
$0 < f + f' < 2$ $\Rightarrow$ $f + f' = 1$

with help of above analysis answer the following questions

अनुच्छेद # 3 (Q.No. 7 to 9)

माना $(7 + 4\sqrt{3})^n = I + f = {}^nC_0.7^n + {}^nC_1.7^{n-1}.(4\sqrt{3})^1 + ...$ ....(i)

जहाँ I तथा f इसके पूर्णांक व भिन्नात्मक भाग है

अर्थात् $0 < f < 1$

अब, $0 < 7 - 4\sqrt{3} < 1 \Rightarrow 0 < (7 - 4\sqrt{3})^n < 1$

मानाकि $(7 - 4\sqrt{3})^n = f' = {}^nC_0.7^n - {}^nC_1.7^{n-1}.(4\sqrt{3})^1 + ...$ ....(ii)

$\Rightarrow 0 < f' < 1$

(i) व (ii) का योग करने पर इससे अपरिमेय पद निरस्त हो जायेगें

$I + f + f' = (7 + 4\sqrt{3})^n + (7 - 4\sqrt{3})^n$

$= 2\,[{}^nC_0\, 7^n + {}^nC_2\, 7^{n-2}\,(4\sqrt{3})^2 + ...]$

$I + f + f'$ = समपूर्णांक $\Rightarrow$ ($f + f'$ एक पूर्णांक होना चाहिए)

$0 < f + f' < 2 \Rightarrow f + f' = 1$

उपरोक्त विश्लेषण के आधार पर निम्न प्रश्नों के उत्तर दीजिए।

7. If $\left(3\sqrt{3} + 5\right)^n = p + f$, where p is an integer and f is a proper fraction, then find the value of $\left(3\sqrt{3} - 5\right)^n$, $n \in N$, is

(A*) $1 - f$, if n is even (B) f, if n is even (C) $1 - f$, if n is odd (D*) f, if n is odd

यदि $\left(3\sqrt{3} + 5\right)^n = p + f$, जहाँ p पूर्णांक है और f भिन्नात्मक भाग है, तो $\left(3\sqrt{3} - 5\right)^n$, $n \in N$ का मान है।

(A*) $1 - f$, यदि n सम है। (B) f, यदि n सम है।

(C) $1 - f$, यदि n विषम है। (D*) f, यदि n विषम है।

Sol. $p + f = (3\sqrt{3} + 5)^n = {}^nC_0(3\sqrt{3})^n 5^0 + {}^nC_1(3\sqrt{3})^{n-1} 5^1 + ...$

$f' = (3\sqrt{3} - 5)^n = {}^nC_0(3\sqrt{3})^n 5^0 - {}^nC_1(3\sqrt{3})^{n-1} 5^1 + ...$

$p + f + f' = 2\,[{}^nC_0(3\sqrt{3})^n + {}^nC_2(3\sqrt{3})^{n-2}5^2 + ...]$

$\Rightarrow p + f + f'$ = even integer सम पूर्णांक (if n is even) (यदि n सम है)

$\Rightarrow f + f' = 1 \Rightarrow f' = 1 - f$

$p + f - f' = 2\,[{}^nC_1(3\sqrt{3})^{n-1}(5) + {}^nC_3(3\sqrt{3})^{n-3}5^3 + ...]$ (if n is odd) (यदि n विषम है)

$\Rightarrow f - f' = 0 \Rightarrow f' = f$

8. If $\left(9 + \sqrt{80}\right)^n = I + f$, where I, n are integers and $0 < f < 1$, then :

(A*) I is an odd integer (B) I is an even integer

(C*) $(I + f)(1 - f) = 1$ (D*) $1 - f = \left(9 - \sqrt{80}\right)^n$

यदि $\left(9 + \sqrt{80}\right)^n = I + f$ जहाँ I, n पूर्णांक हैं और $0 < f < 1$, तो –

(A) I एक विषम पूर्णांक हैं। (B) I एक सम पूर्णांक है।

(C) $(I + f)(1 - f) = 1$ (D) $1 - f = \left(9 - \sqrt{80}\right)^n$

Sol. $\left(9 + \sqrt{80}\right)^n = I + f$

$\left(9 - \sqrt{80}\right)^n = f'$

$2[{}^nC_0\,(9)^n + {}^nC_2\,(9)^{n-2}\,(\sqrt{80})^2 + ...] = I + f + f'$

$\therefore$ I = 2(integer पूर्णांक) – 1 $(\because f + f' = 1)$

$\therefore$ $(I + f)(1 - f) = 1$

9. The integer just above $(\sqrt{3} + 1)^{2n}$ is, for all $n \in N$.

(A*) divisible by 2^n (B*) divisible by 2^{n+1}

(C*) divisible by 8 (D) divisible by 16

$(\sqrt{3} + 1)^{2n}$ से ठीक 1 अधिक पूर्णांक हो, समीकरण $n \in N$ के लिए

(A*) 2^n का भाग देने पर (B*) 2^{n+1} का भाग देने पर

(C*) 8 का भाग देने पर (D) 16 का भाग देने पर

Sol. Let (माना) $(\sqrt{3} + 1)^{2n} = (4 + 2\sqrt{3})^n = 2^n (2 + \sqrt{3})^n = I + f$ ....(i)

where I and f are its integral & fractional parts respectively

जहाँ I तथा f इसके पूर्णांक इसके क्रमशः पूर्णांक व भिन्नात्मक भाग है

$0 < f < 1$.

Now अब $0 < \sqrt{3} - 1 < 1$

$0 < (\sqrt{3} - 1)^{2n} < 1$

Let माना कि $(\sqrt{3} - 1)^{2n} = (4 - 2\sqrt{3})^n = 2^n (2 - \sqrt{3})^n = f'$. ....(ii)

$0 < f' < 1$

adding (i) and (ii) (i) और (ii) को जोड़ने पर

$I + f + f' = (\sqrt{3} + 1)^{2n} + (\sqrt{3} - 1)^{2n}$

$= 2^n [(2 + \sqrt{3})^n + (2\sqrt{3} -)^n] = 2.2^n [{}^nC_0 2^n + {}^nC_2 2^{n-2} (\sqrt{3})^2 + ...]$

$I + f + f' = 2^{n+1} k$ (where k is a positive integer)

$I + f + f' = 2^{n+1} k$ (जहाँ k धनात्मक पूर्णांक है)

$0 < f + f' < 2 \Rightarrow f + f' = 1$

$I + 1 = 2^{n+1} k$.

$I + 1$ is the integer just above $(\sqrt{3} + 1)^{2n}$ and which is divisible by 2^{n+1}.

$I + 1$ ठीक अधिक पूर्णांक है $(\sqrt{3} + 1)^{2n}$ तथा यह 2^{n+1} से विभाजित है।

for से $n = 1$, $(\sqrt{3} + 1)^{2n} = (\sqrt{3} + 1)^2 \Rightarrow I + 1 = 8$

so it is divisible by 8 but not by 16.

यह 8 से विभाजित हो परन्तु 16 से हो।

Exercise-3

✍ Marked questions are recommended for Revision.

✍ चिन्हित प्रश्न दोहराने योग्य प्रश्न है।

PART - I : JEE (ADVANCED) / IIT-JEE PROBLEMS (PREVIOUS YEARS)

भाग - I : JEE (ADVANCED) / IIT-JEE (पिछले वर्षो) के प्रश्न

*** Marked Questions may have more than one correct option.**

*** चिन्हित प्रश्न एक से अधिक सही विकल्प वाले प्रश्न है -**

1. Coefficient of t^{24} in $(1 + t^2)^{12} (1 + t^{12}) (1 + t^{24})$ is: **[IIT-JEE-2003, Scr, (3, – 1), 84]**

$(1 + t^2)^{12} (1 + t^{12}) (1 + t^{24})$ में t^{24} का गुणांक है :

(A) ${}^{12}C_6 + 3$ (B) ${}^{12}C_6 + 1$ (C) ${}^{12}C_6$ (D*) ${}^{12}C_6 + 2$

Sol. $(1 + t^2)^{12} (1 + t^{12} + t^{24} + t^{36}) = (1 + t^{12} + t^{24}) (1 + t^2)^{12}$

coefficient of $t^{24} = {}^{12}C_{12} + {}^{12}C_6 + {}^{12}C_0 = {}^{12}C_6 + 2$

Hindi $(1 + t^2)^{12} (1 + t^{12} + t^{24} + t^{36}) = (1 + t^{12} + t^{24}) (1 + t^2)^{12}$

t^{24} का गुणांक $= {}^{12}C_{12} + {}^{12}C_6 + {}^{12}C_0 = {}^{12}C_6 + 2$

2. Prove that $2^k \binom{n}{0}\binom{n}{k} - 2^{k-1}\binom{n}{1}\binom{n-1}{k-1} + 2^{k-2}\binom{n}{2}\binom{n-2}{k-2} - ...... + (-1)^k \binom{n}{k}\binom{n-k}{0} = \binom{n}{k}$.

[IIT-JEE-2003, Main, (2, 0), 60]

सिद्ध कीजिए कि $2^k \binom{n}{0}\binom{n}{k} - 2^{k-1}\binom{n}{1}\binom{n-1}{k-1} + 2^{k-2}\binom{n}{2}\binom{n-2}{k-2} - ...... + (-1)^k \binom{n}{k}\binom{n-k}{0} = \binom{n}{k}$.

Sol. $S = 2^k\, {}^nC_0.\, {}^nC_k - 2^{k-1}\, {}^nC_1\, {}^{n-1}C_{k-1} + 2^{k-2}\, {}^nC_2\, {}^{n-2}C_{k-2} + ...$

$\Rightarrow S = \sum_{r=0}^{k} (-1)^r\ {}^nC_r\ {}^{n-r}C_{k-r} \cdot 2^{k-r}$ $\qquad \Rightarrow S = \sum_{r=0}^{k} (-1)^r \frac{n!}{r!(n-r)!} \times \frac{(n-r)!.2^{k-r}}{(n-k)!\ (k-r)!}$

$= \sum_{r=0}^{k} (-1)^r\ 2^{k-r} \frac{n!}{k!\ (n-k)!} \times \frac{k!}{r!\ (k-r)!} = 2^k\ {}^nC_k \left(1-\frac{1}{2}\right)^k = {}^nC_k$

3. If ${}^{(n-1)}C_r = (k^2 - 3)\ {}^nC_{r+1,}$ then an interval in which k lies is **[IIT-JEE-2004, Scr, (3, – 1), 84]**

यदि ${}^{(n-1)}C_r = (k^2 - 3)\ {}^nC_{r+1}$ हो, तो k के मान का अन्तराल है – **[Scr, (3, – 1), 84]**

(A) $(2, \infty)$ (B) $(-\infty, -2)$ (C) $\left[-\sqrt{3}, \sqrt{3}\right]$ (D*) $\left(\sqrt{3}, 2\right]$

Sol. ${}^{(n-1)}C_r = (k^2 - 3)\ {}^nC_{r+1}$

or या ${}^{(n-1)}C_{n-(r+1)} = (k^2 - 3)\ {}^nC_{n-(r+1)}$

$1 \ge k^2 - 3 > 0 \quad \Rightarrow k \in [-2, -\sqrt{3}) \cup (\sqrt{3}, 2]$

4. The value of **[IIT-JEE-2005, Scr, (3, – 1), 84]**

$\binom{30}{0}\binom{30}{10} - \binom{30}{1}\binom{30}{11} + \binom{30}{2}\binom{30}{12} - .... + \binom{30}{20}\binom{30}{30}$ is :

(A) $\binom{60}{20}$ (B*) $\binom{30}{10}$ (C) $\binom{30}{15}$ (D) None of these

$\binom{30}{0}\binom{30}{10} - \binom{30}{1}\binom{30}{11} + \binom{30}{2}\binom{30}{12} - .... + \binom{30}{20}\binom{30}{30}$ का मान है– **[Scr, (3, – 1), 84]**

(A) $\binom{60}{20}$ (B) $\binom{30}{10}$ (C) $\binom{30}{15}$ (D) इनमें से कोई नहीं

Sol. $S = {}^{30}C_0\ {}^{30}C_{20} - {}^{30}C_1\ {}^{30}C_{19} + {}^{30}C_2\ {}^{30}C_{18}$................

S = Co-efficient of x^{20} in $(1-x)^{30} (1 + x)^{30}$

S = Co-efficient of x^{20} in $(1-x^2)^{30} = {}^{30}C_{10}$

Hindi. $S = {}^{30}C_0\ {}^{30}C_{20} - {}^{30}C_1\ {}^{30}C_{19} + {}^{30}C_2\ {}^{30}C_{18}$................

$S = (1-x)^{30} (1 + x)^{30}$ में x^{20} का गुणांक

$S = (1-x^2)^{30}$ में x^{20} का गुणांक $= {}^{30}C_{10}$

5. For r = 0, 1, ..., 10, let A_r, B_r and C_r denote, respectively, the coefficient of x^r in the expansions of $(1 + x)^{10}$, $(1 + x)^{20}$ and $(1 + x)^{30}$. Then $\sum_{r=1}^{10} A_r(B_{10}B_r - C_{10}A_r)$ is equal to

मानाकि $r = 0, 1, ..., 10$ के लिए A_r, B_r तथा C_r क्रमशः $(1 + x)^{10}$, $(1 + x)^{20}$ तथा $(1 + x)^{30}$ के प्रसार में x^r के गुणांक हैं।
तो $\sum_{r=1}^{10} A_r(B_{10}B_r - C_{10}A_r)$ का मान निम्न है **[IIT-JEE 2010, Paper-2, (5, –2)/79]**

(A) $B_{10} - C_{10}$ (B) $A_{10}(B^2_{10} - C_{10}A_{10})$ (C) 0 (D*) $C_{10} - B_{10}$

Sol. $B_{10}\sum_{r=1}^{10} A_rB_r - C_{10}\sum_{r=1}^{10}(A_r)^2 = {}^{20}B_{10}({}^{30}C_{20} - 1) - {}^{30}C_{10}({}^{20}C_{10} - 1) = {}^{30}C_{10} - {}^{20}C_{10} = C_{10} - B_{10}$

6. The coefficients of three consecutive terms of $(1 + x)^{n+5}$ are in the ratio 5 : 10 : 14. Then n =
$(1 + x)^{n+5}$ के तीन क्रमागत पदों के गुणांक 5 : 10 : 14 के अनुपात में है, तब n =
Ans. 6

[JEE (Advanced) 2013, Paper-1, (4, – 1)/60]

Sol. ${}^{n+5}C_{r-1} : {}^{n+5}C_r : {}^{n+5}C_{r+1} = 5 : 10 : 14$

$\Rightarrow \quad \frac{{}^{n+5}C_r}{{}^{n+5}C_{r-1}} = \frac{10}{5} \quad \& \quad \frac{{}^{n+5}C_{r+1}}{{}^{n+5}C_r} = \frac{14}{10}$

$\Rightarrow \quad \frac{(n+5) - r + 1}{r} = 2 \quad \& \quad \frac{(n+5) - (r+1) + 1}{r+1} = \frac{7}{5}$

$\Rightarrow \quad \frac{n+6}{r} = 3 \quad \& \quad \frac{n+6}{r+1} = \frac{12}{5} \quad \Rightarrow \quad 3r = \frac{12}{5}(r + 1) \quad \Rightarrow \quad r = 4$

$\therefore \quad n + 6 = 12 \quad \Rightarrow \quad n = 6$

7. Coefficient of x^{11} in the expansion of $(1 + x^2)^4 (1 + x^3)^7 (1 + x^4)^{12}$ is
[JEE (Advanced) 2014, Paper-2, (3, –1)/60]
$(1 + x^2)^4 (1 + x^3)^7 (1 + x^4)^{12}$ विस्तार में (expansion) x^{11} का गुणांक (coefficient) है–
[JEE (Advanced) 2014, Paper-2, (3, –1)/60]

(A) 1051 (B) 1106 (C) 1113 (D) 1120

Ans. **(C)**

Sol. Coefficent of $x^{11} \equiv \frac{(1+x^2)^4(1+x^3)^7(1+x^4)^{12}(1-x^2)^4}{(1-x^2)^4}$

Coefficent of $x^{11} \equiv (1 - x^8)^4 (1 + x^4)^8 (1 + x^3)^7 (1 - x^2)^{-4}$
$= (1 - 4x^8) (1 + x^4)^8 (7x^3 + 35x^9) (1 - x^2)^{-4}$
$= (7x^3 + 35x^9 - 28x^{11}) (1 + x^4)^8 (1 - x^2)^{-4}$
Coefficent of $x^8 = (7x + 35x^6 - 28x^8) (1 + 8x^4 + 28x^8) (1 - x^2)^{-4}$
$= (7 + 35x^6 - 28x^8 + 56x^4 + 196x^8) (1 - x^2)^{-4}$
Coefficent of $t^4 \equiv (7 + 56t^2 + 35t^3 + 168t^4) (1 - t)^{-4}$
$= 7 \cdot {}^7C_3 + 56 \cdot {}^5C_3 + 35 \cdot {}^4C_3 + 168$
$= 245 + 700 + 168 = 1113.$

Hindi $\frac{(1+x^2)^4(1+x^3)^7(1+x^4)^{12}(1-x^2)^4}{(1-x^2)^4}$ में x^{11} का गुणांक

$1 - x^8)^4 (1 + x^4)^8 (1 + x^3)^7 (1 - x^2)^{-4}$ में x^{11} का गुणांक
$= (1 - 4x^8) (1 + x^4)^8 (7x^3 + 35x^9) (1 - x^2)^{-4}$
$= (7x^3 + 35x^9 - 28x^{11}) (1 + x^4)^8 (1 - x^2)^{-4}$
$(7x + 35x^6 - 28x^8) (1 + 8x^4 + 28x^8) (1 - x^2)^{-4}$ में x^8 का गुणांक
$= (7 + 35x^6 - 28x^8 + 56x^4 + 196x^8) (1 - x^2)^{-4}$
$(7 + 56t^2 + 35t^3 + 168t^4) (1 - t)^{-4}$ में t^4 का गुणांक
$= 7 \cdot {}^7C_3 + 56 \cdot {}^5C_3 + 35 \cdot {}^4C_3 + 168$
$= 245 + 700 + 168 = 1113.$

Alterantive : वैकल्पिक हल

$2x + 3y + 4z = 11$
$(x, y, z) = (0, 1, 2)$ ${}^4C_0 \times {}^7C_1 \times {}^{12}C_2$
$(1, 3, 0)$ ${}^4C_1 \times {}^7C_3$
$(2, 1, 1)$ ${}^4C_2 \times {}^7C_1 \times {}^{12}C_1$
$(4, 1, 0)$ ${}^7C_1$
coefficient of x^{11} का गुणांक $= 66 \times 7 + 35 \times 4 + 42 \times 12 + 7$
$= 1113$. **Ans.**

8. The coefficient of x^9 in the expansion of $(1 + x)(1 + x^2)(1 + x^3)......(1 + x^{100})$ is **(Moderate)**
$(1 + x)(1 + x^2)(1 + x^3)......(1 + x^{100})$ के विस्तार में x^9 के गुणांक का मान है
[JEE (Advanced) 2015, P-2 (4, 0) / 80]

Ans. **8**
Sol. 9 = (0, 9) (1, 8), (2, 7), (3, 6), (4, 5) # 5 cases
9 = (1,2,6), (1,3,5), (2, 3, 4) # 3 cases
total = 8

9. Let m be the smallest positive integer such that the coefficient of x^2 in the expansion of $(1 + x)^2 + (1 + x)^3 + ... + (1 + x)^{49} + (1 + mx)^{50}$ is $(3n + 1)\ {}^{51}C_3$ for some positive integer n. Then the value of n is
[JEE (Advanced) 2016, Paper-1, (3, 0)/62]
माना कि m ऐसा न्यूनतम धनात्मक पूर्णांक (smallest positive integer) है कि $(1 + x)^2 + (1 + x)^3 + ... + (1 + x)^{49} + (1 + mx)^{50}$ के विस्तार में x^2 का गुणांक $(3n + 1)\ {}^{51}C_3$ किसी धनात्मक पूर्णांक n के लिए है। तब n का मान है–

Ans. **5**
Sol. Coeff. x^2 का गुणांक
${}^2C_2 + {}^3C_2 + {}^4C_2 + ... + {}^{49}C_2 + {}^{50}C_2\, m^2 = (3n + 1)\ {}^{51}C_3$
${}^3C_3 + {}^3C_2 + {}^4C_2 + ... + {}^{49}C_2 + {}^{50}C_2\, m^2 = (3n + 1)\ {}^{51}C_3$
${}^nC_r + {}^nC_{r-1} = {}^{n+1}C_r \quad \Rightarrow \quad {}^{50}C_3 + {}^{50}C_2 \,.\, m^2 = (3n + 1)\ {}^{51}C_3$
${}^{50}C_3 + {}^{50}C_2 + (m^2 - 1)\ {}^{50}C_2 = 3n. \dfrac{51}{3}. {}^{50}C_2 + {}^{51}C_3 \quad \Rightarrow \quad {}^{51}C_3 + (m^2 - 1)\ {}^{50}C_2 = 51n \,.\, {}^{50}C_2 + {}^{51}C_3$
$m^2 - 1 = 51n \quad \Rightarrow \quad m^2 = 51n + 1$
min value of m^2 for $51n + 1$ is integer for $n = 5$ ($51n + 1$ के पूर्णांक होने के लिए m^2 का न्यूनतम मान $n = 5$)

10. Let $X = ({}^{10}C_1)^2 + 2({}^{10}C_2)^2 + 3({}^{10}C_3)^2 + ... + 10({}^{10}C_{10})^2$ where ${}^{10}C_r$, $r \in \{1, 2, ..., 10\}$ denote binomial coefficients. Then the value of $\dfrac{1}{1430}$ X is _______ . **[JEE (Advanced) 2018, Paper-1, (3, 0)/60]**
माना कि $X = ({}^{10}C_1)^2 + 2({}^{10}C_2)^2 + 3({}^{10}C_3)^2 + ... + 10({}^{10}C_{10})^2$, जहाँ ${}^{10}C_r$, $r \in \{1, 2, ..., 10\}$, द्विपद गुणांकों (binomial coefficients) को दर्शाते हैं। तब $\dfrac{1}{1430}$ X का मान है _______ ।

Ans. **(646)**
Sol. $X = \sum\limits_{r=1}^{10} r.\ {}^{10}C_r . {}^{10}C_r = 10.\sum\limits_{r=1}^{10} {}^9C_{r-1} . {}^{10}C_{10-r} = 10\ .\ {}^{19}C_9$
Now अब $\dfrac{X}{1430} = \dfrac{10.\ {}^{19}C_9}{1430} = \dfrac{{}^{19}C_9}{143} = \dfrac{{}^{19}C_9}{11 \times 13} = \dfrac{19.17.16}{8} = 19 \times 34 = 646$

PART - II : JEE (MAIN) / AIEEE PROBLEMS (PREVIOUS YEARS)

भाग - II : JEE (MAIN) / AIEEE (पिछले वर्षों) के प्रश्न

1. Let $S_1 = \sum_{j=1}^{10} j\ (j-1)\ {}^{10}C_j$, $S_2 = \sum_{j=1}^{10} j\ {}^{10}C_j$ and $S_3 = \sum_{j=1}^{10} j^2\ {}^{10}C_j$. **[AIEEE 2009, (4, –1), 144]**

Statement -1 : $S_3 = 55 \times 2^9$.

Statement -2 : $S_1 = 90 \times 2^8$ and $S_2 = 10 \times 2^8$.

(1) Statement -1 is true, Statement-2 is true ; Statement -2 is not a correct explanation for Statement -1.
(2*) Statement-1 is true, Statement-2 is false.
(3) Statement -1 is false, Statement -2 is true.
(4) Statement -1 is true, Statement -2 is true; Statement-2 is a correct explanation for Statement-1.

माना $S_1 = \sum_{j=1}^{10} j\ (j-1)\ {}^{10}C_j$, $S_2 = \sum_{j=1}^{10} j\ {}^{10}C_j$ तथा $S_3 = \sum_{j=1}^{10} j^2\ {}^{10}C_j$.

प्रकथन -1 : $S_3 = 55 \times 2^9$.

प्रकथन -2 : $S_1 = 90 \times 2^8$ तथा $S_2 = 10 \times 2^8$.

(1) प्रकथन-1 सत्य है, प्रकथन-2 सत्य है ; प्रकथन-2, प्रकथन-1 की सही व्याख्या नहीं है।
(2*) प्रकथन-1 सत्य है, प्रकथन-2 मिथ्या है।
(3) प्रकथन-1 मिथ्या है, प्रकथन-2 सत्य है।
(4) प्रकथन-1 सत्य है, प्रकथन-2 सत्य है ; प्रकथन-2, प्रकथन-1 की सही व्याख्या है।

Sol. $S_1 = \sum_{j=1}^{10} j(j-1)\ .\ \frac{10(10-1)}{j\ (j-1)}\ {}^{8}C_{j-2}$

$\Rightarrow \quad S_1 = 9 \times 10 \sum_{j=2}^{10} {}^{8}C_{j-2} \quad \Rightarrow \quad S_1 = 90\ .\ 2^8$

$S_2 = \sum_{j=1}^{10} j\ .\ \frac{10}{j}\ {}^{9}C_{j-1} = 10.2^9$

$S_3 = \sum_{j=1}^{10} (j(j-1)+j)\ {}^{10}C_j = \sum_{j=1}^{10} j(j-1)\ {}^{10}C_j + \sum_{j=1}^{10} j\ {}^{10}C_j = 90 \sum_{j=2}^{10} {}^{8}C_{j-2} + 10\sum_{j=1}^{10} {}^{9}C_{j-1}$

$= 90 \times 2^8 + 10 \times \sum_{j=2}^{10} {}^{8}C_{j-2}\ \ 2^9 = (45 + 10)\ .\ 2^9 = (45 + 10)\ .\ 2^9 = 55.2^9$

so statement-1 is true and statement 2 is false.
इसलिए कथन-1 सत्य है तथा कथन - 2 असत्य है।
Hence correct option is (2)
अतः सही विकल्प (2) है।

2. The coefficient of x^7 in the expansion of $(1 - x - x^2 + x^3)^6$ is : **[AIEEE 2011, (4, –1), 120]**
$(1 - x - x^2 + x^3)^6$ के प्रसार में x^7 का गुणांक है :

(1) 144 (2) – 132 (3*) – 144 (4) 132

Sol. **(3)**
$(1 - x - x^2 + x^3)^6$
$(1 - x)^6\ (1 - x^2)^6$
$({}^6C_0 - {}^6C_1 x^1 + {}^6C_2 x^2 - {}^6C_3x^3 + {}^6C_4x^4 - {}^6C_5x^5 + {}^6C_6x^6)\ ({}^6C_0 - {}^6C_1x^2 + {}^6C_2x^4 - {}^6C_3x^6 + {}^6C_4x^8 + ... + {}^6C_6x^{12})$
Now coefficient of $x^7 = {}^6C_1{}^6C_3 - {}^6C_3{}^6C_2 + {}^6C_5{}^6C_1$
$= 6 \times 20 - 20 \times 15 + 36$
$= 120 - 300 + 36$
$= 156 - 300 = -144$ **Ans.**

Hindi $(1 - x - x^2 + x^3)^6$
$(1 - x)^6\ (1 - x^2)^6$
$({}^6C_0 - {}^6C_1 x^1 + {}^6C_2 x^2 - {}^6C_3x^3 + {}^6C_4x^4 - {}^6C_5x^5 + {}^6C_6x^6)\ ({}^6C_0 - {}^6C_1x^2 + {}^6C_2x^4 - {}^6C_3x^6 + {}^6C_4x^8 + ... + {}^6C_6x^{12})$
x^7 का गुणांक $= {}^6C_1{}^6C_3 - {}^6C_3{}^6C_2 + {}^6C_5{}^6C_1$
$= 6 \times 20 - 20 \times 15 + 36$

$= 120 - 300 + 36$
$= 156 - 300$
$= -144$ **Ans.**

3. If n is a positive integer, then $\left(\sqrt{3}+1\right)^{2n} - \left(\sqrt{3}-1\right)^{2n}$ is : **[AIEEE 2012, (4, –1), 120]**

(1*) an irrational number (2) an odd positive integer
(3) an even positive integer (4) a rational number other than positive integers

यदि n एक धनपूर्णांक है, तो $\left(\sqrt{3}+1\right)^{2n} - \left(\sqrt{3}-1\right)^{2n}$ – :

(1*) एक अपरिमेय संख्या है। (2) एक विषम धनपूर्णांक है।
(3) एक सम धनपूर्णांक है। (4) धनपूर्णांकों को छोड़ कर एक परिमेय संख्या है।

Sol. **Ans. (1)**
$(\sqrt{3} + 1)^{2n} - (\sqrt{3} - 1)^{2n}$
$=2[{}^{2n}C_1(\sqrt{3})^{2n-1} + {}^{2n}C_3(\sqrt{3})^{2n-3} + {}^{2n}C_5(\sqrt{3})^{2n-5} + ...]$
= which is an irrational number

Hindi. $(\sqrt{3} + 1)^{2n} - (\sqrt{3} - 1)^{2n}$
$=2[{}^{2n}C_1(\sqrt{3})^{2n-1} + {}^{2n}C_3(\sqrt{3})^{2n-3} + {}^{2n}C_5(\sqrt{3})^{2n-5} + ...]$
= जो कि एक अपरिमेय संख्या है।

4. The term independent of x in expansion of $\left(\frac{x+1}{x^{2/3}-x^{1/3}+1} - \frac{x-1}{x-x^{1/2}}\right)^{10}$ is :**[AIEEE - 2013, (4, –1),120]**

$\left(\frac{x+1}{x^{2/3}-x^{1/3}+1} - \frac{x-1}{x-x^{1/2}}\right)^{10}$ के प्रसार में x से स्वतंत्र पद है : **[AIEEE - 2013, (4, –1),120]**

(1) 4 (2) 120 (3*) 210 (4) 310

Sol. (3)
$$\left((x^{1/3}+1) - \left(\frac{\sqrt{x}+1}{\sqrt{x}}\right)\right)^{10}$$
$(x^{1/3} - x^{-1/2})^{10}$
$T_{r+1} = {}^{10}c_r\,(x^{1/3})^{10-r}\,(-x^{-1/2})^r$
$$\frac{10-r}{3} - \frac{r}{2} = 0 \Rightarrow \quad 20 - 2r - 3r = 0$$
$\Rightarrow \quad r = 4$
$$T_5 = {}^{10}C_4 = \frac{10 \times 9 \times 8 \times 7}{4 \times 3 \times 2 \times 1} = 210$$

5. If the coefficients of x^3 and x^4 in the expansion of $(1 + ax + bx^2)(1 - 2x)^{18}$ in powers of x are both zero, then (a, b) is equal to **[Binomial Theorem]** **[JEE(Main) 2014, (4, – 1), 120]**

यदि $(1 + ax + bx^2)(1 - 2x)^{18}$ के x की घातों में प्रसार में x^3 तथा x^4 दोनों के गुणांक शून्य हैं, तो (a, b) बराबर है–
[Binomial Theorem] **[JEE(Main) 2014, (4, – 1), 120]**

(1) $\left(14, \frac{272}{3}\right)$ (2*) $\left(16, \frac{272}{3}\right)$ (3) $\left(16, \frac{251}{3}\right)$ (4) $\left(14, \frac{251}{3}\right)$

Sol. **Ans. (2)**
$(1 + ax + bx^2)(1 - 2x)^{18}$
coeff of x^3 का गुणांक $= {}^{18}C_3(-2)^3 + a.(-2)^2 . {}^{18}C_2 + b.(-2). {}^{18}C_1 = 0$
coeff of x^4 का गुणांक $\equiv {}^{18}C_4(-2)^4 + a(-2)^3 . {}^{18}C_3 + b(-2)^2 . {}^{18}C_2 = 0$

$\Rightarrow$ 51a – 3b = 544 and और 32a – 3b = 240

Subtracting we get घटाने पर प्राप्त होता है a = 16

$\Rightarrow$ $b = \frac{272}{3}$

6. The sum of coefficients of integral powers of x in the binomial expansion of $(1 - 2\sqrt{x})^{50}$ is

$(1 - 2\sqrt{x})^{50}$ के द्विपद प्रसार में x की पूर्णांकीय घातो के गुणांकों का योग है :**[JEE(Main) 2015, (4, – 1), 120]**

(1) $\frac{1}{2}(3^{50} + 1)$ (2) $\frac{1}{2}(3^{50})$ (3) $\frac{1}{2}(3^{50} - 1)$ (4) $\frac{1}{2}(2^{50} + 1)$

Ans. (1)

Sol. $\left(1-2\sqrt{x}\right)^{50} = C_0 - C_1\ 2\sqrt{x} + C_2\left(2\sqrt{x}\right)^2 + \ldots\ldots + C_{50}\left(2\sqrt{x}\right)^{50}$

$\left(1+2\sqrt{x}\right)^{50} = C_0 + C_1\left(2\sqrt{x}\right) + C_2\left(2\sqrt{x}\right)^2 + \ldots\ldots + C_{50}\left(2\sqrt{x}\right)^{50}$

Put x = 1 रखने पर

$\therefore$ $\frac{1+3^{50}}{2} = C_0 + C_2(2)^2 + \ldots$

7. If the number of terms in the expansion of $\left(1-\frac{2}{x}+\frac{4}{x^2}\right)^n$, $x \neq 0$, is 28, then the sum of the coefficients of all the terms in this expansion, is **[JEE(Main) 2016, (4, – 1), 120]**

यदि $\left(1-\frac{2}{x}+\frac{4}{x^2}\right)^n$, $x \neq 0$ के प्रसार में पदों की संख्या 28 है, तो इस प्रसार में आने वाले सभी पदों के गुणांकों का योग है–

(1) 2187 (2) 243 (3) 729 (4) 64

Ans. (3) or Bonus

Sol. Theortically the number of terms are 2N + 1 (i.e. odd) But As the number of terms being odd hence considering that number clubbing of terms is done hence the solutions follwos :

Number of terms = ${}^{n+2}C_2 = 28$ $\therefore$ n = 6

sum of cofficient = $3^n = 3^6 = 729$

put x = 1

सिद्धान्तः पदों की संख्या 2N + 1 है (अर्थात् विाम) परन्तु जैसा कि पदों की संख्या विषम है अतः पदों के मिश्रण के अनुसार हल लिखने पर।

पदो की संख्या = ${}^{n+2}C_2 = 28$ $\therefore$ n = 6

गुणांको का योग = $3^n = 3^6 = 729$

x = 1 रखने पर

8. The value of $({}^{21}C_1 - {}^{10}C_1) + ({}^{21}C_2 - {}^{10}C_2) + ({}^{21}C_3 - {}^{10}C_3) + ({}^{21}C_4 - {}^{10}C_4) + \ldots\ldots + ({}^{21}C_{10} - {}^{10}C_{10})$ is

$({}^{21}C_1 - {}^{10}C_1) + ({}^{21}C_2 - {}^{10}C_2) + ({}^{21}C_3 - {}^{10}C_3) + ({}^{21}C_4 - {}^{10}C_4) + \ldots\ldots + ({}^{21}C_{10} - {}^{10}C_{10})$ का मान है–

[JEE(Main) 2017, (4, – 1), 120]

(1) $2^{21} - 2^{11}$ (2) $2^{21} - 2^{10}$ (3) $2^{20} - 2^9$ (4) $2^{20} - 2^{10}$

Ans. (4)

Sol. $({}^{21}C_1 + {}^{21}C_2 + {}^{21}C_3 + \ldots\ldots {}^{21}C_{10}) - ({}^{10}C_1 + {}^{10}C_2 + {}^{10}C_3 + \ldots\ldots {}^{10}C_{10}) = S_1 - S_2$

$S_1 = {}^{21}C_1 + {}^{21}C_2 + {}^{21}C_3 + \ldots\ldots {}^{21}C_{10}$

$S_1 = \frac{1}{2}({}^{21}C_1 + {}^{21}C_2 + \ldots\ldots + {}^{21}C_{20}) = \frac{1}{2}({}^{21}C_0 + {}^{21}C_1 + {}^{21}C_2 + \ldots\ldots + {}^{21}C_{20} + {}^{21}C_{21} - 2)$

$S_1 = 2^{20} - 1$

$S_2 = ({}^{10}C_1 + {}^{10}C_2 + {}^{10}C_3 + \ldots\ldots {}^{10}C_{10}) = 2^{10} - 1$

Therefore इसलिए, $S_1 - S_2 = 2^{20} - 2^{10}$

9. The sum of the co-efficients of all odd degree terms in the expansion of $\left(x+\sqrt{x^3-1}\right)^5+\left(x-\sqrt{x^3-1}\right)^5$, $(x > 1)$ is : **[JEE(Main) 2018, (4, – 1), 120]**

$\left(x+\sqrt{x^3-1}\right)^5+\left(x-\sqrt{x^3-1}\right)^5$, $(x > 1)$ के प्रसार में सभी विषम घातों वाले पदों के गुणांको का योग है :

(1) 1 (2) 2 (3) –1 (4) 0

Sol. **(2)**

$\left(x+\sqrt{x^3-1}\right)^5+\left(x-\sqrt{x^3-1}\right)^5$

$= (T_1 + T_2 + T_3 + T_4 + T_5 + T_6) + (T_1 - T_2 + T_3 - T_4 + T_5 - T_6)$

$= 2(T_1 + T_3 + T_5)$

$= 2({}^5C_0(x)^5 + {}^5C_2(x)^3\left(\sqrt{x^3-1}\right)^2 + {}^5C_4(x)^1\left(\sqrt{x^3-1}\right)^4$

$= 2(x^5 + 10x^3(x^3-1) + 5x(x^6 + 1 - 2x^3)]$

$= 2(x^5 + 10x^6 - 10x^3 + 5x^7 + 5x - 10x^4)$

$= 2(5x^7 + 10x^6 + x^5 - 10x^4 - 10x^3 + 5x)$

sum of odd degree terms विषम घात के पदों का योगफल $= 10 + 2 - 20 + 10 = 2$

10. If the fractional part of the number $\frac{2^{403}}{15}$ is $\frac{k}{15}$, then k is equal to :

[JEE(Main) 2019, Online (09-01-19),P-1 (4, – 1), 120]

यदि संख्या $\frac{2^{403}}{15}$ का भिन्नात्मक भाग (fractional part) $\frac{k}{15}$ है तो k बराबर है–

(1) 14 (2) 8 (3) 6 (4) 4

Ans. **(2)**

Sol. $\left\{\frac{2^{203}}{15}\right\}$

$8.2^{200} \Rightarrow 8.16^{50} = 8(1 + 15)^{50} = 8(1 + 15)$

hence remainder is 8. (अतः शेषफल 8 है।)

11. If $\sum_{i=1}^{20}\left(\frac{{}^{20}C_{i-1}}{{}^{20}C_i+{}^{20}C_{i-1}}\right)^3 = \frac{k}{21}$, then k equals : **[JEE(Main) 2019, Online (10-01-19),P-1 (4, – 1), 120]**

यदि $\sum_{i=1}^{20}\left(\frac{{}^{20}C_{i-1}}{{}^{20}C_i+{}^{20}C_{i-1}}\right)^3 = \frac{k}{21}$, तो k बराबर है:

(1) 50 (2) 400 (3) 200 (4) 100

Ans. **(4)**

Sol. $\sum_{i=1}^{20}\left(\frac{{}^{20}C_{i-1}}{{}^{20}C_i+{}^{20}C_{i-1}}\right)^3$

Now अब $\frac{{}^{20}C_{i-1}}{{}^{20}C_i+{}^{20}C_{i-1}} = \frac{{}^{20}C_{i-1}}{{}^{21}C_i} = \frac{i}{21}$

Let given sum be S, so माना दिया गया योग S है, तब

$S = \sum_{i=1}^{20}\frac{(i)^3}{21^3} = \frac{1}{(21)^3}\left(\frac{20.21}{2}\right)^2 = \frac{100}{21}$

Given दिया गया है कि $S = \frac{k}{21} \Rightarrow k = 100$

12. If $\sum_{r=0}^{25}\left\{{}^{50}C_r.{}^{50-r}C_{25-r}\right\} = K\left({}^{50}C_{25}\right)$, then K is equal to :

यदि $\sum_{r=0}^{25}\left\{{}^{50}C_r . {}^{50-r}C_{25-r}\right\} = K\left({}^{50}C_{25}\right)$ है, तो K बराबर है :**[JEE(Main) 2019, Online (10-01-19),P-2 (4, – 1), 120]**

(1) 2^{25} (2) $2^{25} - 1$ (3) $(25)^2$ (4) 2^{24}

Ans. **(1)**

Sol. $\sum_{r=0}^{25} {}^{50}C_r \, {}^{50-r}C_{25-r}$

$= \sum_{r=0}^{25} \frac{50!}{r!(50-r)!}\frac{(50-r)!}{(25-r)!(25)!}$

$= \sum_{r=0}^{25} \frac{50! \; 25!}{r!(25-r)!(25)!25!}$

$= {}^{50}c_{25} \sum_{r=0}^{25} {}^{25}c_r = {}^{50}c_{25} \times 2^{25} = k\left({}^{50}c_{25}\right)$

$\Rightarrow k = 2^{25}$

13. Let $S_n = 1 + q + q^2 + \ldots + q^n$ and $T_n = 1 + \left(\frac{q+1}{2}\right) + \left(\frac{q+1}{2}\right)^2 + \ldots + \left(\frac{q+1}{2}\right)^n$.

where q is a real number and $q \neq 1$. If ${}^{101}C_1 + {}^{101}C_2 . S_1 + \ldots + {}^{101}C_{101} . S_{100} = \alpha T_{100}$ then α is equal to

माना $S_n = 1 + q + q^2 + \ldots + q^n$ तथा $T_n = 1 + \left(\frac{q+1}{2}\right) + \left(\frac{q+1}{2}\right)^2 + \ldots + \left(\frac{q+1}{2}\right)^n$. जहाँ q एक वास्तविक संख्या है तथा $q \neq 1$ यदि ${}^{101}C_1 + {}^{101}C_2 . S_1 + \ldots + {}^{101}C_{101} . S_{100} = \alpha T_{100}$ तो α बराबर है –

[JEE(Main) 2019, Online (11-01-19),P-2 (4, – 1), 120]

(1) 200 (2) 2^{99} (3) 2^{100} (4) 202

Ans. **(3)**

Sol. $\sum_{r=1}^{101} {}^{101}C_r S_{r-1} = \sum_{r=1}^{101} {}^{101}C_r \frac{q^r - 1}{q-1} = \frac{1}{q-1}\left(\sum_{r=1}^{101} {}^{101}C_r q^r - \sum_{r=1}^{101} {}^{101}C_r\right) = \frac{1}{q-1}\left((1+q)^{101} - 1 - 2^{101} + 1\right)$

$\Rightarrow \quad \alpha \frac{\left(\frac{q+1}{2}\right)^{101} - 1}{\frac{q+1}{2} - 1} = \frac{1}{q-1}\left((1+q)^{101} - 2^{101}\right)$

$\Rightarrow \quad \frac{\alpha}{2^{100}}\left(\frac{(1+q)^{101} - 2^{101}}{q-1}\right) = \frac{1}{q-1}\left((1+q)^{101} - 2^{101}\right)$

$\Rightarrow$ Hence अतः $\alpha = 2^{100}$

High Level Problems (HLP)

SUBJECTIVE QUESTIONS

विषयात्मक प्रश्न (SUBJECTIVE QUESTIONS)

1. Find the coefficient of x^{49} in

$\left(x+\frac{C_1}{C_0}\right)\left(x+2^2\frac{C_2}{C_1}\right)\left(x+3^2\frac{C_3}{C_2}\right)........\left(x+50^2\frac{C_{50}}{C_{49}}\right)$ where $C_r = {}^{50}C_r$

Ans. 22100

$\left(x+\frac{C_1}{C_0}\right)\left(x+2^2\frac{C_2}{C_1}\right)\left(x+3^2\frac{C_3}{C_2}\right)........\left(x+50^2\frac{C_{50}}{C_{49}}\right)$ में x^{49} का गुणांक ज्ञात कीजिए। (जहाँ $C_r = {}^{50}C_r$)

Sol. Coeff. of $x^{49} = \left(\frac{C_1}{C_0}+2^2\frac{C_2}{C_1}+3^2\frac{C_3}{C_2}+............+50^2\frac{C_{50}}{C_{49}}\right) = \sum_{r=1}^{50} r^2\frac{{}^{50}C_r}{{}^{50}C_{r-1}} = \sum_{r=1}^{50} r^2\left(\frac{50-r+1}{r}\right)$

$= \sum_{r=1}^{50} r(51-r) = \frac{51\times50\times51}{2}-\frac{50\times51\times101}{6} = 22100$

2. The expression, $\left(\sqrt{2x^2+1}+\sqrt{2x^2-1}\right)^6+\left(\frac{2}{\sqrt{2x^2+1}+\sqrt{2x^2-1}}\right)^6$ is a polynomial of degree

व्यंजक $\left(\sqrt{2x^2+1}+\sqrt{2x^2-1}\right)^6+\left(\frac{2}{\sqrt{2x^2+1}+\sqrt{2x^2-1}}\right)^6$ किस घात का एक बहुपद है–

Ans. 6

Sol. $\left(\sqrt{2x^2+1}+\sqrt{2x^2-1}\right)^6+\left(\frac{2\left(\sqrt{2x^2+1}-\sqrt{2x^2-1}\right)}{(2x^2+1-2x^2+1)}\right)^6$

$= \left(\sqrt{2x^2+1}+\sqrt{2x^2-1}\right)^6+\left(\sqrt{2x^2+1}-\sqrt{2x^2-1}\right)^6$

$= 2\left({}^6C_0(2x^2+1)^3+{}^6C_2(2x^2+1)^2(2x^2-1)+{}^6C_4(2x^2+1)(2x^2-1)^2+{}^6C_6(2x^2-1)^3\right)$

clearly '6'

Hindi $\left(\sqrt{2x^2+1}+\sqrt{2x^2-1}\right)^6+\left(\frac{2\left(\sqrt{2x^2+1}-\sqrt{2x^2-1}\right)}{(2x^2+1-2x^2+1)}\right)^6$

$= \left(\sqrt{2x^2+1}+\sqrt{2x^2-1}\right)^6+\left(\sqrt{2x^2+1}-\sqrt{2x^2-1}\right)^6$

$= 2\left({}^6C_0(2x^2+1)^3+{}^6C_2(2x^2+1)^2(2x^2-1)+{}^6C_4(2x^2+1)(2x^2-1)^2+{}^6C_6(2x^2-1)^3\right)$

स्पष्टया '6'

3. Find the co-efficient of x^5 in the expansion of $(1 + x^2)^5 (1 + x)^4$.

$(1 + x^2)^5 (1 + x)^4$ के विस्तार में x^5 का गुणांक ज्ञात कीजिए।

Ans. 60

Sol. Co-efficient of x^5 in $(1 + x^2)^5 (1+x)^4 = {}^4C_1 \cdot {}^5C_2 + {}^4C_3 \cdot {}^5C_1 = 40 + 20 = 60$

Hindi $(1 + x^2)^5 (1+x)^4$ में x^5 का गुणांक $= {}^4C_1 \cdot {}^5C_2 + {}^4C_3 \cdot {}^5C_1 = 40 + 20 = 60$

4. Prove that the co-efficient of x^{15} in $(1 + x + x^3 + x^4)^n$ is $\sum_{r=0}^{5} {}^nC_{15-3r}\,{}^nC_r$.

सिद्ध कीजिए कि $(1 + x + x^3 + x^4)^n$ के विस्तार में x^{15} का गुणांक $\sum_{r=0}^{5} {}^nC_{15-3r}\,{}^nC_r$ है

Sol. $(1 + x + x^3 + x^4)^n = [(1 + x)(1 + x^3)]^n = (1 + x)^n (1 + x^3)^n$

power of x in $(1 + x^3)^n$ expansion is multiple of 3

so possible cases to get x^{15} are :-

(9n) $(1 + x)^n$	9n$(1 + x^3)^n$		
$\Downarrow$	$\Downarrow$		
${}^nC_{15}\,x^{15}$	${}^nC_0\,(x^3)^0$	=	${}^nC_{15}\,.\,{}^nC_0\,x^{15}$
${}^nC_{12}\,x^{12}$	${}^nC_1\,(x^3)^1$	=	${}^nC_{12}\,.\,{}^nC_1\,x^{15}$
${}^nC_{12}\,x^9$	${}^nC_2\,(x^3)^2$	=	${}^nC_9\,.\,{}^nC_2\,x^{15}$
${}^nC_6\,x^6$	${}^nC_3\,(x^3)^3$	=	${}^nC_6\,.\,{}^nC_3\,x^{15}$
${}^nC_3\,x^3$	${}^nC_4\,(x^3)^4$	=	${}^nC_3\,.\,{}^nC_4\,x^{15}$
${}^nC_0\,x^0$	${}^nC_5\,(x^3)^5$	=	${}^nC_0\,.\,{}^nC_5\,x^{15}$

$\therefore$ coefficient of x^{15} is

$= {}^nC_{15}\,.\,{}^nC_0 + {}^nC_{12}\,.\,{}^nC_1 + {}^nC_9\,.\,{}^nC_2 + {}^nC_6\,.\,{}^nC_3 + {}^nC_3\,.\,{}^nC_4 + {}^nC_0\,.\,{}^nC_5$

$= \sum_{r_2=0}^{5} {}^nC_{15-3r}.{}^nC_r$ hence prooved

Hindi $(1 + x + x^3 + x^4)^n = [(1 + x)(1 + x^3)]^n = (1 + x)^n (1 + x^3)^n$

$(1 + x^3)^n$ में x की घात, 3 का गुणज है।

x^{15} की संभाविय स्थिति

(9n) $(1 + x)^n$	9n$(1 + x^3)^n$		
$\Downarrow$	$\Downarrow$		
${}^nC_{15}\,x^{15}$	${}^nC_0\,(x^3)^0$	=	${}^nC_{15}\,.\,{}^nC_0\,x^{15}$
${}^nC_{12}\,x^{12}$	${}^nC_1\,(x^3)^1$	=	${}^nC_{12}\,.\,{}^nC_1\,x^{15}$
${}^nC_{12}\,x^9$	${}^nC_2\,(x^3)^2$	=	${}^nC_9\,.\,{}^nC_2\,x^{15}$
${}^nC_6\,x^6$	${}^nC_3\,(x^3)^3$	=	${}^nC_6\,.\,{}^nC_3\,x^{15}$
${}^nC_3\,x^3$	${}^nC_4\,(x^3)^4$	=	${}^nC_3\,.\,{}^nC_4\,x^{15}$
${}^nC_0\,x^0$	${}^nC_5\,(x^3)^5$	=	${}^nC_0\,.\,{}^nC_5\,x^{15}$

$\therefore$ x^{15} का गुणांक है।

$= {}^nC_{15}\,.\,{}^nC_0 + {}^nC_{12}\,.\,{}^nC_1 + {}^nC_9\,.\,{}^nC_2 + {}^nC_6\,.\,{}^nC_3 + {}^nC_3\,.\,{}^nC_4 + {}^nC_0\,.\,{}^nC_5$

$= \sum_{r_2=0}^{5} {}^nC_{15-3r}.{}^nC_r$ अतः सिद्व हुआ

5. If n is even natural and coefficient of x^r in the expansion of $\frac{(1+x)^n}{1-x}$ is 2^n, ($|x| < 1$), then prove that $r \ge n$

यदि n सम प्राकृत संख्या है तथा $\frac{(1+x)^n}{1-x}$ के विस्तार में x^r का गुणांक 2^n ($|x| < 1$) है तब सिद्ध कीजिए $r \ge n$

Sol. $y = (1 - x)^{-1} (1 + x)^n$

$y = (1 + x + x^2 + \ldots \infty) (1 + x)^n$

$y = (1 + x)^n + x(1 + x)^n + x^2 (1 + x)^n + \ldots.$

Co-efficient of x^r =

${}^nC_r + {}^nC_{r-1} + \ldots.\ {}^nC_0 = 2^n$

$r \geq n$ (As ${}^nC_{n+1}) = 0$

Hindi $y = (1 - x)^{-1} (1 + x)^n$

$y = (1 + x + x^2 + \ldots \infty)(1 + x)^n$

$y = (1 + x)^n + x(1 + x)^n + x^2 (1 + x)^n + \ldots$

x^r का गुणांक =

${}^nC_r + {}^nC_{r-1} + \ldots\ {}^nC_0 = 2^n$

$r \geq n$ (As ${}^nC_{n+1}) = 0$

6. Find the coefficient of x^n in polynomial $(x + {}^{2n+1}C_0)(x + {}^{2n+1}C_1)\ldots\ldots(x + {}^{2n+1}C_n)$.

बहुपद $(x + {}^{2n+1}C_0)(x + {}^{2n+1}C_1)\ldots\ldots(x + {}^{2n+1}C_n)$ में x^n का गुणांक ज्ञात कीजिए।

Ans. 2^{2n}

Sol. Co-efficient of $x^n = {}^{2n+1}C_0 + {}^{2n+1}C_1 + \ldots + {}^{2n+1}C_n = 2^{2n}$

Hindi x^n का गुणांक $= {}^{2n+1}C_0 + {}^{2n+1}C_1 + \ldots + {}^{2n+1}C_n = 2^{2n}$

7. Find the value of $\sum_{r=1}^{n} \left(\sum_{p=0}^{r-1} {}^nC_r\, {}^rC_p\, 2^p \right)$.

$\sum_{r=1}^{n} \left(\sum_{p=0}^{r-1} {}^nC_r\, {}^rC_p\, 2^p \right)$ का मान ज्ञात कीजिए।

Ans. $4^n - 3^n$

Sol. $$\sum_{r=1}^{n} \left(\sum_{p=0}^{r-1} {}^nC_r\ {}^rC_p 2^p \right) = \sum_{r=1}^{n} {}^nC_r \sum_{p=0}^{r-1} {}^rC_p . 2^p = \sum_{r=1}^{n} {}^nC_r\ [{}^rC_0 + {}^rC_1 . 2 + \ldots\ldots + {}^rC_{r-1} 2^{r-1}]$$

$$= \sum_{r=1}^{n} {}^nC_r\ (3^r - 2^r) = 4^n - 3^n$$

Comprehension (Q-8 to Q.10)
अनुच्छेद

For k, n $\in$ N, we define

$B(k, n) = 1.2.3\ldots\ldots\ k + 2.3.4\ldots\ldots(k+1) + \ldots\ldots + n(n + 1)\ldots\ldots(n + k - 1)$, $S_0(n) = n$ and $S_k(n) = 1^k + 2^k + \ldots\ldots + n^k$.

To obtain value B(k, n), we rewrite B(k, n) as follows

$$B(k,n) = k!\left[{}^kC_k + {}^{k+1}C_k + {}^{k+2}C_k + \ldots\ldots + {}^{n+k-1}C_k\right] = k!\left({}^{n+k}C_{k+1}\right)$$

$$= \frac{n(n+1)\ldots\ldots(n+k)}{k+1}$$

where ${}^nC_k = \dfrac{n!}{k!\ (n-k)!}$

k, n $\in$ N के लिए परिभाषित किया जाता है कि

$B(k, n) = 1.2.3\ldots\ldots\ k + 2.3.4\ldots\ldots(k+1) + \ldots\ldots + n(n + 1)\ldots\ldots(n + k - 1)$, $S_0(n) = n$ एवं $S_k(n) = 1^k + 2^k + \ldots\ldots + n^k$.

B(k, n) का मान ज्ञात करने के लिए B(k, n) को निम्न प्रकार पुनः लिखने पर

$$B(k,n) = k!\left[{}^kC_k + {}^{k+1}C_k + {}^{k+2}C_k + \ldots\ldots + {}^{n+k-1}C_k\right] = k!\left({}^{n+k}C_{k+1}\right)$$

$$= \frac{n(n+1)\ldots\ldots(n+k)}{k+1} \text{ जहाँ } {}^nC_k = \frac{n!}{k!\ (n-k)!}$$

8. Prove that $S_2(n) + S_1(n) = B(2, n)$
सिद्ध कीजिए $S_2(n) + S_1(n) = B(2, n)$

Sol. $S_2(n) + S_1(n) = \sum n^2 + \sum n$
$= \sum n(n+1)$
$= 1.2 + 2.3 + 3.4 + \text{.........} + n(n + 1)$
$= B(2, n)$

9. Prove that सिद्ध कीजिए $S_3(n) + 3S_2(n) = B(3, n) - 2B(1, n)$

Sol. $S_3(n) + 3S_2(n) + 2S_1(n) - 2\, S_1(n)$
$= \sum n^3 + 3\sum n^2 + 2\sum n - 2\sum n$
$= \sum n(n+1)(n+2) - 2\sum n$
$= B(3, n) - 2\, B(1, n)$

10. If $(1 + x)^p = 1 + {}^pC_1\, x + {}^pC_2x^2 + \text{..........} + {}^pC_p\, x^p,\ p \in N$, then show that ${}^{k+1}C_1\, S_k(n) + {}^{k+1}C_2\, S_{k-1}(n) + \text{.........} + {}^{k+1}C_k\, S_1(n) + {}^{k+1}C_{k+1}\, S_0(n) = (n + 1)^{k+1} - 1$
यदि $(1 + x)^p = 1 + {}^pC_1\, x + {}^pC_2x^2 + \text{..........} + {}^pC_p\, x^p,\ p \in N$ तब दर्शाइये कि ${}^{k+1}C_1\, S_k(n) + {}^{k+1}C_2\, S_{k-1}(n) + \text{.........} + {}^{k+1}C_k\, S_1(n) + {}^{k+1}C_{k+1}\, S_0(n) = (n + 1)^{k+1} - 1$

Sol. $(1 + x)^{k+1} = {}^{k+1}C_{k+1} + {}^{k+1}C_k\, x + {}^{k+1}C_{k-1}\, x^2 + {}^{k+1}C_{k-2}\, x^3 + \text{.........} + {}^{k+1}C_0\, x^{k+1}$
Put $x = 1, 2, \text{......}, n$ रखने पर
$2^{k+1} = {}^{k+1}C_{k+1} + {}^{k+1}C_k\, .\, 1 + {}^{k+1}C_{k-1}.1^2 + {}^{k+1}C_{k-2}\, .\, 1^3 + \text{.........} + {}^{k+1}C_0\, .\, 1^{k+1}$
$3^{k+1} = {}^{k+1}C_{k+1} + {}^{k+1}C_k\, .\, 2 + {}^{k+1}C_{k-1}\, .2^2 + \text{........} + {}^{k+1}C_0 .\, 2^{k+1}$
$\vdots$
$(1 + n)^{k+1} = {}^{k+1}C_{k+1} + {}^{k+1}C_k\, .\, n + {}^{k+1}C_{k-1},\, n^2 + \text{........} + {}^{k+1}C_0\, n^{k+1}$
$2^{k+1} + 3^{k+1} + \text{...........} + (1 + n)^{k+1} = {}^{k+1}C_{k+1}\, S_0(n) + {}^{k+1}C_k\, S_1(n) + {}^{k+1}C_{k-1}\, S_2(n) + \text{...........} + {}^{k+1}C_0\, S_{(k+1)}(n)$
$2^{k+1} + 3^{k+1} + \text{............} + (n+1)^{k+1} = {}^{k+1}C_{k+1}\, S_0(n) + {}^{k+1}C_k\, S_1(n) + \text{........} + {}^{k+1}C_1\, S_k(n) + 1^{k+1} + 2^{k+1} + 3^{k+1} + \text{..........} + n^{k+1}$
So $(n + 1)^{k+1} - 1$

Hindi $(1 + x)^{k+1} = {}^{k+1}C_{k+1} + {}^{k+1}C_k\, x + {}^{k+1}C_{k-1}\, x^2 + {}^{k+1}C_{k-2}\, x^3 + \text{.........} + {}^{k+1}C_0\, x^{k+1}$
$x = 1, 2, \text{......}, n$ रखने पर
$2^{k+1} = {}^{k+1}C_{k+1} + {}^{k+1}C_k\, .\, 1 + {}^{k+1}C_{k-1}.1^2 + {}^{k+1}C_{k-2}\, .\, 1^3 + \text{.........} + {}^{k+1}C_0\, .\, 1^{k+1}$
$3^{k+1} = {}^{k+1}C_{k+1} + {}^{k+1}C_k\, .\, 2 + {}^{k+1}C_{k-1}\, .2^2 + \text{........} + {}^{k+1}C_0 .\, 2^{k+1}$
$\vdots$
$(1 + n)^{k+1} = {}^{k+1}C_{k+1} + {}^{k+1}C_k\, .\, n + {}^{k+1}C_{k-1},\, n^2 + \text{........} + {}^{k+1}C_0\, n^{k+1}$
$2^{k+1} + 3^{k+1} + \text{...........} + (1 + n)^{k+1} = {}^{k+1}C_{k+1}\, S_0(n) + {}^{k+1}C_k\, S_1(n) + {}^{k+1}C_{k-1}\, S_2(n) + \text{...........} + {}^{k+1}C_0\, S_{(k+1)}(n)$
$2^{k+1} + 3^{k+1} + \text{............} + (n+1)^{k+1} = {}^{k+1}C_{k+1}\, S_0(n) + {}^{k+1}C_k\, S_1(n) + \text{........} + {}^{k+1}C_1\, S_k(n) + 1^{k+1} + 2^{k+1} + 3^{k+1} + \text{..........} + n^{k+1}$
अतः $(n + 1)^{k+1} - 1$

11. Show that $25^n - 20^n - 8^n + 3^n,\ n \in I^+$ is divisible by 85.
प्रदर्शित कीजिए कि $25^n - 20^n - 8^n + 3^n,\ n \in I^+$, 85 से भाज्य है।

Sol. $85 = 17 \times 5$ (Both are prime number)
$25^n = (20 + 5)^n$
and $8^n = (5 + 3)^n$
So clearly $(20 + 5)^n - 20^n - (5 + 3)^n + 3^n$ is divisible by 5
Also $(17 + 8)^n - 8^n - (17 + 3)^n + 3^n$ is divisible by 17
So expression is divisible by 85

Hindi $85 = 17 \times 5$ (दोनो अभाज्य संख्याएँ है।)

$25^n = (20 + 5)^n$ तथा $8^n = (5 + 3)^n$

अतः स्पष्टतया $(20 + 5)^n - 20^n - (5 + 3)^n + 3^n$, 5 से भाज्य है

पुनः $(17 + 8)^n - 8^n - (17 + 3)^n + 3^n$, 17 से भाज्य है अतः व्यंजक 85 से भाज्य है।

12. Prove that ${}^nC_1\ ({}^nC_2)^2\ ({}^nC_3)^3........({}^nC_n)^n \le \left(\dfrac{2^n}{n+1}\right)^{{}^{n+1}C_2}$.

सिद्ध कीजिए कि ${}^nC_1\ ({}^nC_2)^2\ ({}^nC_3)^3........({}^nC_n)^n \le \left(\dfrac{2^n}{n+1}\right)^{{}^{n+1}C_2}$.

Sol. A.M. $\ge$ G.M

$$\frac{{}^nC_1 + 2.{}^nC_2 + 3.{}^nC_3 + ... + n.{}^nC_n}{1+2+3+...........+n} \ge \sqrt[\frac{n(n+1)}{2}]{{}^nC_1\left({}^nC_2\right)^2........\left({}^nC_n\right)^n}$$

$$= \frac{n.2^{n-1}\cdot 2}{n(n+1)} \ge \sqrt[\frac{n(n+1)}{2}]{{}^nC_1.\left({}^nC_2\right)^2.........\left({}^nC_n\right)^n}$$

$${}^nC_1\left({}^nC_2\right)^2.......\left({}^nC_n\right)^n \le \left(\frac{2^n}{n+1}\right)^{\frac{n(n+1)}{2}} \qquad \left(\text{Also}\,\frac{n(n+1)}{2} = {}^{n+1}C_2\right)$$

13. If p is nearly equal to q and n > 1, show that $\dfrac{(n+1)\ p+(n-1)q}{(n-1)p+(n+1)q} = \left(\dfrac{p}{q}\right)^{1/n}$. Hence find the approximate value of $\left(\dfrac{99}{101}\right)^{1/6}$.

यदि p, q के लगभग बराबर है तथा n > 1, प्रदर्शित कीजिए कि $\dfrac{(n+1)\ p+(n-1)q}{(n-1)p+(n+1)q} = \left(\dfrac{p}{q}\right)^{1/n}$. इसकी सहायता से $\left(\dfrac{99}{101}\right)^{1/6}$ का निकटतम मान ज्ञात कीजिए।

Ans. $\dfrac{1198}{1202}$

Sol. Let p = q + h (say), where h is small that its square and higher powers may be neglected. then -

$$\frac{(n+1)\ p\ +\ (n-1)\ q}{(n-1)p\ +\ (n+1)\ q} = \frac{(n+1)\ (q+h)\ +\ (n-1)\ q}{(n-1)\ (q+h)\ +\ (n+1)\ q} \qquad (\because\ p = q + h)$$

$$= \frac{2nq+(n+1)\ h}{2nq\ +\ (n-1)\ h} = \left(1+\left(\frac{n+1}{2nq}\right)\ h\right)\left(1+\left(\frac{n-1}{2nq}\right)\ h\right)^{-1} = 1 + \frac{h}{nq} = \left(1+\frac{h}{q}\right)^{1/n} = \left(\frac{p}{q}\right)^{1/n}$$

put p = 99 , q = 101 and n = 6

$$\frac{(6+1)\ \times 99+(6-1)\times 101}{(6-1)\ \times 99+(6+1)\ \times\ 101} = \left(\frac{99}{101}\right)^{1/6} = \frac{1198}{1202}$$

Hindi मानाकि p = q + h (माना), जहाँ h इतना छोटा है कि इसके वर्ग तथा अन्य बड़ी घातों को नगण्य मान सकते है, तो

$$\frac{(n+1)\ p\ +\ (n-1)\ q}{(n-1)p\ +\ (n+1)\ q} = \frac{(n+1)\ (q+h)\ +\ (n-1)\ q}{(n-1)\ (q+h)\ +\ (n+1)\ q} \qquad (\because p = q + h)$$

$$= \frac{2nq+(n+1)\ h}{2nq\ +\ (n-1)\ h} = \left(1+\left(\frac{n+1}{2nq}\right)\ h\right)\left(1+\left(\frac{n-1}{2nq}\right)\ h\right)^{-1} = 1 + \frac{h}{nq} = \left(1+\frac{h}{q}\right)^{1/n} = \left(\frac{p}{q}\right)^{1/n}$$

p = 99 , q = 101 तथा n = 6 रखने पर

$$\frac{(6+1)\times 99+(6-1)\times 101}{(6-1)\times 99+(6+1)\times 101}=\left(\frac{99}{101}\right)^{1/6}=\frac{1198}{1202}$$

14. If $(18x^2 + 12x + 4)^n = a_0 + a_1x + a_2x^2 + ... + a_{2n}x^{2n}$, then prove that

$a_r = 2^n 3^r \left({}^{2n}C_r + {}^nC_1\ {}^{2n-2}C_r + {}^nC_2\ {}^{2n-4}C_r + ...\right)$

यदि $(18x^2 + 12x + 4)^n = a_0 + a_1x + a_2x^2 + ... + a_{2n}x^{2n}$, तब सिद्ध कीजिए

$a_r = 2^n 3^r \left({}^{2n}C_r + {}^nC_1\ {}^{2n-2}C_r + {}^nC_2\ {}^{2n-4}C_r + ...\right)$

Sol. a_r is the coefficient of x^r in R.H.S.

$(18x^2 + 12x + 4)^n = 2^n (1+(1+3x)^2)^n$

$= 2^n\left({}^nC_0(1+3x)^{2n} + {}^nC_1\ (1+3x)^{2n-2} + {}^nC_2\ (1+3x)^{2n-4} + ...\right)$

$= 2^n\left({}^nC_0 3^r\ {}^{2n}C_r + {}^nC_1 3^r\ {}^{2n-2}C_r + {}^nC_2 3^r\ {}^{2n-4}C_r + ...\right)$

Hindi. a_r, R.H.S. में x^r का गुणांक है।

$(18x^2 + 12x + 4)^n = 2^n (1+(1+3x)^2)^n$

$= 2^n\left({}^nC_0(1+3x)^{2n} + {}^nC_1\ (1+3x)^{2n-2} + {}^nC_2\ (1+3x)^{2n-4} + ...\right)$

$= 2^n\left({}^nC_0 3^r\ {}^{2n}C_r + {}^nC_1 3^r\ {}^{2n-2}C_r + {}^nC_2 3^r\ {}^{2n-4}C_r + ...\right)$

15. Prove that $1^2 . C_0 + 2^2 . C_1 + 3^2 . C_2 + 4^2 . C_3 + ... + (n+1)^2 C_n = 2^{n-2}(n+1)(n+4)$.

Sol. $(1 + x)^n = C_0 + C_1x + C_2x^2 + ... + C_nx^n$

multiply by x and then differentiate

$(1 + x)^n + x . n (1 + x)^{n-1} = C_0 + 2.C_1x + 3.C_2 x^2 + ... + (n + 1) . C_n x^n$

again multiply by x and then differentiate

$(1 + x)^n + nx (1 + x)^{n-1} + 2nx (1 + x)^{n-1} + n (n - 1) x^2 (1 + x)^{n-2} = C_0 + 2^2 C_1x + 3^2 C_2x^2 + ...+ (n+1)^2 C_nx^n$

put x = 1

then $S = 2^n + n. 2^{n-1} + 2n. 2^{n-1} + n (n - 1) 2^{n-2}$

$= 2^{n-2} [4 + 2n + 4n + n^2 - n]$

$= 2^{n-2} (n + 1) (n + 4)$

Hindi. $(1 + x)^n = C_0 + C_1x + C_2x^2 + ... + C_nx^n$

x से गुणा करने के बाद अवकलन करने पर

$(1 + x)^n + x . n (1 + x)^{n-1} = C_0 + 2.C_1x + 3.C_2 x^2 + ... + (n + 1) . C_n x^n$

दुबारा x से गुणा करके अवकलन करने पर

$(1 + x)^n + nx (1 + x)^{n-1} + 2nx (1 + x)^{n-1} + n (n - 1) x^2 (1 + x)^{n-2} = C_0 + 2^2 C_1x + 3^2 C_2x^2 + ...+ (n+1)^2 C_nx^n$

x = 1 रखने पर

तब $S = 2^n + n. 2^{n-1} + 2n. 2^{n-1} + n (n - 1) 2^{n-2}$

$= 2^{n-2} [4 + 2n + 4n + n^2 - n]$

$= 2^{n-2} (n + 1) (n + 4)$.

16. If $(1 - x)^{-n} = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + ...$, find the value of, $a_0 + a_1 + a_2 + ... + a_n$.

$(1 - x)^{-n} = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + ...$, $a_0 + a_1 + a_2 + ... + a_n$ का मान ज्ञात करो–

Ans. $\frac{(2n)!}{(n!)^2}$

Sol. $(1 - x)^{-n} = a_0 + a_1x + a_2x^2 + ... a_nx^n = \sum_{r=0}^{n} {}^{n+r-1}C_r x^r$

$a_0 + a_1 + ... + a_n = {}^{n-1}C_0 + {}^{n}C_1 + {}^{n+1}C_2 + ... + {}^{2n-1}C_n$
$= {}^{n-1}C_{n-1} + {}^{n}C_{n-1} + {}^{n+1}C_{n-1} + ... + {}^{2n-1}C_{n-1}$
$= {}^{n}C_n + {}^{n}C_{n-1} + {}^{n+1}C_{n-1} + ... + {}^{2n-1}C_{n-1} = {}^{2n}C_n$ $\{{}^{n}C_r + {}^{n}C_{r-1} = {}^{n+1}C_r\}$

17. Find the remainder when $32^{32^{32}}$ is divided by 7.
$32^{32^{32}}$ को 7 से भाग देने पर शेषफल ज्ञात करो।
Ans. 4
Sol. $32^{32} = (2^5)^{32} = 2^{160}$
$(3-1)^{160} = 3\lambda + 1$
$32^{32^{32}} = (2^5)^{(3\lambda+1)} = 2^{(15\lambda+3)+2} = 4 \,.\, (2^3)^{(5\lambda+1)} = 4(7+1)^{\beta} = 4(7\mu + 1)$
$\therefore$ remainder is 4
Hindi. $32^{32} = (2^5)^{32} = 2^{160}$
$(3-1)^{160} = 3\lambda + 1$
$32^{32^{32}} = (2^5)^{(3\lambda+1)} = 2^{(15\lambda+3)+2} = 4 \,.\, (2^3)^{(5\lambda+1)} = 4(7+1)^{\beta} = 4(7\mu + 1)$
$\therefore$ शेषफल 4 है।

18. If n is an integer greater than 1, show that : $a - {}^{n}C_1(a-1) + {}^{n}C_2(a-2) - ..... + (-1)^n (a-n) = 0$.
यदि n (> 1) एक पूर्णांक है तब प्रदर्शित कीजिए : $a - {}^{n}C_1(a-1) + {}^{n}C_2(a-2) - ..... + (-1)^n (a-n) = 0$.
Sol. $S = a\,[{}^{n}C_0 - {}^{n}C_1 + {}^{n}C_2 ....+ (-1)^n\, {}^{n}C_n] + {}^{n}C_1 - 2.\, {}^{n}C_2 + ... + (-1)^{n+1} n\, {}^{n}C_n$
$(1-x)^n = {}^{n}C_0 - {}^{n}C_1 x + {}^{n}C_2 x^2 .... + (-1)^n .\, {}^{n}C_n\, x^n$
$n(1-x)^{n-1} = -\,{}^{n}C_1 + 2.\, {}^{n}C_2 x + ... + (-1)^n\, n\, {}^{n}C_n\, x^{n-1}$
put x = 1
then S = 0
Hindi. $S = a\,[{}^{n}C_0 - {}^{n}C_1 + {}^{n}C_2 ....+ (-1)^n\, {}^{n}C_n] + {}^{n}C_1 - 2.\, {}^{n}C_2 + ... + (-1)^{n+1} n\, {}^{n}C_n$
$(1-x)^n = {}^{n}C_0 - {}^{n}C_1 x + {}^{n}C_2 x^2 .... + (-1)^n .\, {}^{n}C_n\, x^n$
$n(1-x)^{n-1} = -\,{}^{n}C_1 + 2.\, {}^{n}C_2 x + ... + (-1)^n\, n\, {}^{n}C_n\, x^{n-1}$
x = 1 रखने पर
तब S = 0

19. If $(1+x)^n = p_0 + p_1 x + p_2 x^2 + p_3 x^3 +.......$, then prove that :
यदि $(1+x)^n = p_0 + p_1 x + p_2 x^2 + p_3 x^3 +.......$, तब सिद्ध करो कि
(a) $p_0 - p_2 + p_4 -....... = 2^{n/2} \cos \dfrac{n\pi}{4}$ (b) $p_1 - p_3 + p_5 -....... = 2^{n/2} \sin \dfrac{n\pi}{4}$
Sol. $(1+x)^n = p_0 + p_1x + p_2x^2 ....$
$(1-x)^n = p_0 - p_1x + p_2x^2 ....$
$(1+x)^n + (1-x)^n = 2[p_0 + p_2 x^2 + p_4x^4 + ...]$
Put x = i
then $p_0 - p_2 + p_4 ....= \dfrac{(1+i)^n + (1-i)^n}{2} = 2^{n/2} \cos \dfrac{n\pi}{4}$
and $(1+x)^n - (1-x)^n = 2\,[p_1x + p_3x^3 + ...]$
or $p_1 - p_3 + p_5 .... = \dfrac{(1+i)^n - (1-i)^n}{2i} = 2^{n/2} \sin \dfrac{n\pi}{4}$
Hindi. $(1+x)^n = p_0 + p_1x + p_2x^2 ....$
$(1-x)^n = p_0 - p_1x + p_2x^2 ....$
$(1+x)^n + (1-x)^n = 2[p_0 + p_2 x^2 + p_4x^4 + ...]$
x = i रखने पर

तब $p_0 - p_2 + p_4$ ....= $\frac{(1+i)^n + (1-i)^n}{2} = 2^{n/2} \cos\frac{n\pi}{4}$

और $(1 + x)^n - (1 - x)^n = 2\,[p_1x + p_3x^3 + ...]$

या $p_1 - p_3 + p_5$ .... = $\frac{(1+i)^n - (1-i)^n}{2i} = 2^{n/2} \sin\frac{n\pi}{4}$

20. Show that if the greatest term in the expansion of $(1 + x)^{2n}$ has also the greatest co-efficient, then ' x ' lies between, $\frac{n}{n+1}$ & $\frac{n+1}{n}$.

प्रदर्शित करो कि यदि $(1 + x)^{2n}$ के प्रसार में अधिकतम पद का गुणांक भी अधिकतम है, तो ' x ' का मान $\frac{n}{n+1}$ और $\frac{n+1}{n}$ के बीच में है।

Sol. Middle term has greatest co-efficient in this case so r = n

मध्य पद का गुणांक महत्तम होता है। अतः इस स्थिति में r = n है

$r = \left[\frac{2n+1}{1+|x|}\right] \Rightarrow \frac{2n+1}{1+|x|} - 1 < n < \frac{2n+1}{1+|x|} \qquad \Rightarrow \qquad \frac{2n+1}{n+1} - 1 < |x|$ and तथा $|x| < 1 + \frac{1}{n}$

$\Rightarrow \qquad \frac{n}{n+1} < x < \frac{n+1}{n}$

21. Prove that if ' p ' is a prime number greater than 2, then $\left[(2+\sqrt{5})^p\right] - 2^{p+1}$ is divisible by p, where [.] denotes greatest integer function.

सिद्ध करो कि यदि ' p ', 2 से बड़ी एक अभाज्य संख्या है, तो $\left[(2+\sqrt{5})^p\right] - 2^{p+1}$, p से विभाजित होगा, जहाँ [.] महत्तम पूर्णांक फलन है।

Sol. $\left[(2+\sqrt{5})^p\right] - 2^{p+1}$

Let $(\sqrt{5}+2)^p = I + f$ so $\quad [(\sqrt{5}+2)^p] = I$, where I is an integer and $f \in (0, 1)$

$(\sqrt{5}-2)^p = f' \in (0, 1)$

$2[{}^pC_0\, 2^p + {}^pC_2 2^{p-2} (\sqrt{5})^2 + ...] = I + f - f'$

$\Rightarrow f' - f = 0 \qquad \Rightarrow f = f' \quad \Rightarrow [(2+\sqrt{5})^p] - 2^{p+1} = \quad 2[{}^pC_0 2^p + {}^pC_2 2^{p-2}.5 + ...] - 2^{p+1}$

$= \ {}^pC_2.2^{p-1}.5 + {}^pC_4 2^{p-3}.5^2 +$ ....

This is always divisible by p because for a prime number p, ${}^pC_r(1 < r < p)$ is always divisible by p.

Hindi $\left[(2+\sqrt{5})^p\right] - 2^{p+1}$

माना $(\sqrt{5}+2)^p = I + f$ इसलिए $\quad [(\sqrt{5}+2)^p] = I$, जहाँ I एक पूर्णांक है तथा $f \in (0, 1)$

$(\sqrt{5}-2)^p = f' \in (0, 1)$

$2[{}^pC_0\, 2^p + {}^pC_2 2^{p-2} (\sqrt{5})^2 + ...] = I + f - f'$

$\Rightarrow f' - f = 0 \qquad \Rightarrow f = f' \Rightarrow [(2+\sqrt{5})^p] - 2^{p+1}$

$= \quad 2[{}^pC_0 2^p + {}^pC_2 2^{p-2}.5 + ...] - 2^{p+1}$

$= \ {}^pC_2.2^{p-1}.5 + {}^pC_4 2^{p-3}.5^2 +$ ....

यह हमेशा p से भाजित है क्योंकि एक अभाज्य संख्या p हेतु ${}^pC_r(1 < r < p)$ सदैव p से भाजित है।

22. If $\sum_{r=0}^{n} (-1)^r.\,{}^nC_r \left[\frac{1}{2^r} + \frac{3^r}{2^{2r}} + \frac{7^r}{2^{3r}} + ... \text{ to m terms}\right] = k\left(1 - \frac{1}{2^{mn}}\right)$, then find the value of k.

यदि $\sum_{r=0}^{n} (-1)^r.\,{}^nC_r \left[\frac{1}{2^r} + \frac{3^r}{2^{2r}} + \frac{7^r}{2^{3r}} + ... \text{ m पदों तक}\right] = k\left(1 - \frac{1}{2^{mn}}\right)$ हो, तो k का मान ज्ञात कीजिए।

Ans. $\frac{1}{2^n - 1}$

Sol. $\sum_{r=0}^{n} (-1)^r.\,{}^nC_r \left[\frac{1}{2^r} + \left(\frac{3}{4}\right)^r + \left(\frac{7}{8}\right)^r + ...\text{m terms}\right]$

$= \sum_{r=0}^{n} \left[(-1)^r\,{}^nC_r \frac{1}{2^r} + (-1)^r\,{}^nC_r \left(\frac{3}{4}\right)^r + ...\text{m terms}\right]$

$= [\,{}^nC_0 - {}^nC_1 \left(\frac{1}{2}\right) + {}^nC_2 \left(\frac{1}{2}\right)^2 + ...] +$

$[\,{}^nC_0 - {}^nC_1 + {}^nC_2 + ...] + ...\text{m terms}$

$= \left(1 - \frac{1}{2}\right)^n + \left(1 - \frac{3}{4}\right)^n + \left(1 - \frac{7}{8}\right)^n + ...\text{m terms} = \frac{1}{2^n}\left(\frac{1 - \frac{1}{2^{mn}}}{1 - \frac{1}{2^n}}\right) = \frac{1}{2^n - 1}\left(1 - \frac{1}{2^{mn}}\right)$

Hindi. $\sum_{r=0}^{n} (-1)^r.\,{}^nC_r = \left[\frac{1}{2^r} + \left(\frac{3}{4}\right)^r + \left(\frac{7}{8}\right)^r + ...\text{m terms}\right]$

$= \left[(-1)^r\,{}^nC_r \frac{1}{2^r} + (-1)^r\,{}^nC_r \left(\frac{3}{4}\right)^r + ...\text{m terms}\right]$

$= [\,{}^nC_0 - {}^nC_1 \left(\frac{1}{2}\right) + {}^nC_2 \left(\frac{1}{2}\right)^2 + ...] +$

$[\,{}^nC_0 - {}^nC_1 + {}^nC_2 + ...] + ...$m पद

$= \left(1 - \frac{1}{2}\right)^n + \left(1 - \frac{3}{4}\right)^n + \left(1 - \frac{7}{8}\right)^n + ...$m पद

$\frac{1}{2^n}\left(\frac{1 - \frac{1}{2^{mn}}}{1 - \frac{1}{2^n}}\right) = \frac{1}{2^n - 1}\left(1 - \frac{1}{2^{mn}}\right)$

23. Given $s_n = 1 + q + q^2 + ... + q^n$ & $S_n = 1 + \frac{q+1}{2} + \left(\frac{q+1}{2}\right)^2 + ... + \left(\frac{q+1}{2}\right)^n$, $q \neq 1$,

prove that ${}^{n+1}C_1 + {}^{n+1}C_2.s_1 + {}^{n+1}C_3.s_2 + ... + {}^{n+1}C_{n+1}.s_n = 2^n.\,S_n.$

यदि $s_n = 1 + q + q^2 + ... + q^n$ तथा $S_n = 1 + \frac{q+1}{2} + \left(\frac{q+1}{2}\right)^2 + ... + \left(\frac{q+1}{2}\right)^n$, $q \neq 1$ हो, तो

सिद्ध करो कि ${}^{n+1}C_1 + {}^{n+1}C_2.s_1 + {}^{n+1}C_3.s_2 + ... + {}^{n+1}C_{n+1}.s_n = 2^n.\,S_n.$

Sol. $s_n = \frac{q^{n+1} - 1}{q - 1}$ and और $S_n = \frac{\left(\frac{q+1}{2}\right)^{n+1} - 1}{\frac{q+1}{2} - 1}$

$^{n+1}C_1 + {}^{n+1}C_2 s_1 + ...+ {}^{n+1}C_{n+1}\, s_n$

$= {}^{n+1}C_1 + {}^{n+1}C_2\left(\frac{q^2-1}{q-1}\right) + ...+ {}^{n+1}C_{n+1}\ \frac{q^{n+1}-1}{q-1}$

$= \frac{1}{q-1}\ [{}^{n+1}C_1 q + {}^{n+1}C_2 q^2 + ...+ {}^{n+1}C_{n+1} q^{n+1} - {}^{n+1}C_1 - {}^{n+1}C_2 -.... - {}^{n+1}C_{n+1}]$

$= \frac{1}{q-1}\ [(1+q)^{n+1} - 1 - 2^{n+1} + 1] = \frac{1}{q-1}\ [(1+q)^{n+1} - 2^{n+1}] = \left(\frac{\left(\frac{q+1}{2}\right)^{n+1} - 1}{\frac{q-1}{2}}\right) .\ 2^n\ = 2^n\ S_n$

24. If $(1+x)^{15} = C_0 + C_1.\ x + C_2.\ x^2 + ... + C_{15}.\ x^{15}$, then find the value of : $C_2 + 2C_3 + 3C_4 + ... + 14C_{15}$

यदि $(1+x)^{15} = C_0 + C_1.\ x + C_2.\ x^2 + ... + C_{15}.\ x^{15}$ हो, तो $C_2 + 2C_3 + 3C_4 + ... + 14C_{15}$ का मान ज्ञात करो।

Ans. 212993

Sol. $(1 + x)^{15} = C_0 + C_1x + ...+ C_{15}x^{15}$

Divide by x & then differentiating both side

$\frac{(1+x)^{15}}{x} = \frac{C_0}{x} + C_1 + C_2x + C_3x^2 + ...+ C_{15}x^{14}$

$.\ \frac{1}{x}\, 15(1 + x)^{14} - \frac{(1+x)^{15}}{x^2} = -\frac{C_0}{x^2} + C_2 + ...+ 14\, C_{15}x^{13}$

Put x = 1 then $C_2 + 2C_3 + ...+ 14C_{15} = 15.2^{14} - 2^{15} + 1$

Hindi. $(1 + x)^{15} = C_0 + C_1x + ...+ C_{15}x^{15}$

x से भाग देकर दोनों पक्षों का अवकलन करने पर

$\frac{(1+x)^{15}}{x} = \frac{C_0}{x} + C_1 + C_2x + C_3x^2 + ...+ C_{15}x^{14}$

$.\ \frac{1}{x}\, 15(1 + x)^{14} - \frac{(1+x)^{15}}{x^2} = -\frac{C_0}{x^2} + C_2 + ...+ 14\, C_{15}x^{13}$

x = 1 रखने पर $C_2 + 2C_3 + ...+ 14C_{15} = 15.2^{14} - 2^{15} + 1$

25. Prove that, $\frac{1}{2}\ {}^nC_1 - \frac{2}{3}\,{}^nC_2 + \frac{3}{4}\,{}^nC_3 - \frac{4}{5}\,{}^nC_4 + ... + \frac{(-1)^{n+1}\, n}{n+1}\ .\ {}^nC_n = \frac{1}{n+1}$

सिद्ध कीजिए कि : $\frac{1}{2}\ {}^nC_1 - \frac{2}{3}\,{}^nC_2 + \frac{3}{4}\,{}^nC_3 - \frac{4}{5}\,{}^nC_4 + ... + \frac{(-1)^{n+1}\, n}{n+1}\ .\ {}^nC_n = \frac{1}{n+1}$

Sol. $(1+x)^n = C_0 + C_1x + C_2x^2 + ...+ C_nx^n$

$\Rightarrow n(1 + x)^{n-1} = C_1 + 2.C_2x + 3.C_3x^2 + ...+ n.C_nx^{n-1}$

Now multiply by x & integrate from 0 to x

$n(1 + x)^{n-1}.\ x = C_1x + 2C_2x^2 + ...+ nC_nx^n$

$\Rightarrow \frac{C_1}{2}x^2 + \frac{2C_2x^3}{3} + \frac{3C_3x^4}{4} + ...+ \frac{n}{n+1}C_nx^{n+1} = x(1+x)^n - \frac{(1+x)^{n+1} - 1}{n+1}$

Putting x = –1

$\frac{C_1}{2} - \frac{2}{3}C_2 + \frac{3}{4}C_3 + ... = \frac{1}{n+1}$

Hindi. $(1+x)^n = C_0 + C_1x + C_2x^2 + ...+ C_nx^n$

$\Rightarrow n(1 + x)^{n-1} = C_1 + 2.C_2x + 3.C_3x^2 + ...+ n.C_nx^{n-1}$

x से गुणा करके 0 से x सीमाओं में समाकलन करने पर

$n(1 + x)^{n-1}.\ x = C_1x + 2\, C_2x^2 + ...+ nC_nx^n$

$$\Rightarrow \frac{C_1}{2}x^2 + \frac{2C_2x^3}{3} + \frac{3C_3x^4}{4} + ... + \frac{n}{n+1}C_n x^{n+1} = x(1+x)^n - \frac{(1+x)^{n+1}-1}{n+1}$$

$x = -1$ रखने पर

$$\frac{C_1}{2} - \frac{2}{3}C_2 + \frac{3}{4}C_3 + ... = \frac{1}{n+1}$$

26. Prove that $\sum_{r=0}^{n} r^2\ {}^nC_r\, p^r q^{n-r} = npq + n^2p^2$, if $p + q = 1$.

सिद्ध करो कि $\sum_{r=0}^{n} r^2\ {}^nC_r\, p^r q^{n-r} = npq + n^2p^2$ होगा, जबकि $p + q = 1$ हो।

Sol. $\sum_{r=0}^{n} r^2\ {}^nC_r\ p^r q^{n-r} = n\ .\ \sum_{r=0}^{n} r\ .{}^{n-1}C_{r-1} p^r q^{n-r}$

$$= n.\left[\sum_{r=1}^{n}(r-1)\ .\ {}^{n-1}C_{r-1}p^rq^{n-r} + \sum_{r=1}^{n} {}^{n-1}C_{r-1}.\ \ p^rq^{n-r}\right]$$

$$= n\left[(n-1)p^2\sum_{r=2}^{n}{}^{n-2}C_{r-2}p^{r-2}q^{n-r} + p\sum_{r=1}^{n}{}^{n-1}C_{r-1}.\ \ p^{r-1}q^{n-r}\right]$$

$= n[(n-1)p^2(p+q)^{n-2} + p(p+q)^{n-1}]$

$= n[np^2 - p^2 + p] = n^2p^2 - np^2 + pn = n^2p^2 + npq$

27. Prove that : $(n-1)^2.\ C_1 + (n-3)^2.\ C_3 + (n-5)^2.\ C_5 + ... = n(n+1)2^{n-3}$

सिद्ध कीजिए कि : $(n-1)^2.\ C_1 + (n-3)^2.\ C_3 + (n-5)^2.\ C_5 + ... = n(n+1)2^{n-3}$

Sol. $(n-1)^2\ {}^nC_1 + (n-3)^2\ {}^nC_3 + (n-5)^2\ {}^nC_5 + ...$

$= n^2({}^nC_1 + {}^nC_3 + {}^nC_5 + ...) - 2n\,({}^nC_1 + 3\,{}^nC_3 + 5\,{}^nC_5 + ...) + ({}^nC_1 + 9\,{}^nC_3 + 25\,{}^nC_5 + ...)$

$= n^2\ .\ 2^{n-1} - 2n^2({}^{n-1}C_0 + {}^{n-1}C_2 + {}^{n-1}C_4 + ...) + n({}^{n-1}C_0 + 3\,{}^{n-1}C_2 + 5\,{}^{n-1}C_4 + ...)$

$= n^2\ .\ 2^{n-1} - 2n^2\ .\ (2^{n-2}) + n({}^{n-1}C_0 + {}^{n-1}C_2 + {}^{n-1}C_4 + ...) + n(2\,{}^{n-1}C_2 + 4\ {}^{n-1}C_4 + 6\ {}^{n-1}C_6 + ...)$

$= n^2\ .\ 2^{n-1} - n^2\ .\ 2^{n-1} + n\ .\ 2^{n-2} + n(n-1)\ ({}^{n-2}C_1 + {}^{n-2}C_3 + {}^{n-2}C_5 + ...)$

$= n\ .\ 2^{n-2} + n(n-1)\ 2^{n-3}$

$= n(n+1)\ 2^{n-3}$.

28. Prove that ${}^nC_r + 2\ {}^{n+1}C_r + 3\ {}^{n+2}C_r + ... + (n+1)\ {}^{2n}C_r = {}^nC_{r+2} + (n+1)\ {}^{2n+1}C_{r+1} - {}^{2n+1}C_{r+2}$

सिद्ध करो ${}^nC_r + 2\ {}^{n+1}C_r + 3\ {}^{n+2}C_r + ... + (n+1)\ {}^{2n}C_r = {}^nC_{r+2} + (n+1)\ {}^{2n+1}C_{r+1} - {}^{2n+1}C_{r+2}$

Sol. Let ${}^nC_r + 2.\ {}^{n+1}C_r + 3.\ {}^{n+2}C_r + ... = S$

S = co-efficient of x^r in $(1+x)^n + 2.(1+x)^{n+1} + 3.(1+x)^{n+2} + ...$

Let $S' = (1+x)^n + 2(1+x)^{n+1} + 3(1+x)^{n+2} + ...+ (n+1)(1+x)^{2n}$ .... (1)

$(1+x)\ S' = (1+x)^{n+1} + 2(1+x)^{n+2} + ...+ (n+1)(1+x)^{2n+1}$ .... (2)

(1) – (2) :

$-x\ S' = (1+x)^n + (1+x)^{n+1} + ...+ (1+x)^{2n} - (n+1)\ \ (1+x)^{2n+1}$

$$\Rightarrow -x\ S' = (1+x)^n\left[\frac{(1+x)^{n+1}-1}{x}\right] - (n+1)(1+x)^{2n+1} \quad \Rightarrow S' = \frac{(1+x)^{2n+1}-(1+x)^n}{-x^2} + \frac{(n+1)(1+x)^{2n+1}}{x}$$

Now S = co-efficient of x^r in S'

$= -\ {}^{2n+1}C_{r+2} + {}^nC_{r+2} + (n+1)\ {}^{2n+1}C_{r+1}$

Hindi. माना ${}^nC_r + 2.\ {}^{n+1}C_r + 3.\ {}^{n+2}C_r + ... = S$

$S = (1+x)^n + 2.(1+x)^{n+1} + 3.(1+x)^{n+2} + ...$ में x^r का गुणांक

माना $S' = (1+x)^n + 2(1+x)^{n+1} + 3(1+x)^{n+2} + ...+ (n+1)(1+x)^{2n}$ .... (1)

$(1+x) S' = (1+x)^{n+1} + 2(1+x)^{n+2} + ...+ (n+1)(1+x)^{2n+1}$ .... (2)

(1) – (2) :

$-x S' = (1+x)^n + (1+x)^{n+1} + ...+ (1+x)^{2n} - (n+1) \quad (1+x)^{2n+1}$

$\Rightarrow -x S' = (1+x)^n \left[\dfrac{(1+x)^{n+1}-1}{x}\right] - (n+1)(1+x)^{2n+1} \quad \Rightarrow S' = \dfrac{(1+x)^{2n+1}-(1+x)^n}{-x^2} + \dfrac{(n+1)(1+x)^{2n+1}}{x}$

अब $S = S'$ में x^r का गुणांक

$= -{}^{2n+1}C_{r+2} + {}^nC_{r+2} + (n+1)\ {}^{2n+1}C_{r+1}$

29. Show that, $\sqrt{3} = 1 + \dfrac{1}{3} + \dfrac{1}{3}.\dfrac{3}{6} + \dfrac{1}{3}.\dfrac{3}{6}.\dfrac{5}{9} + \dfrac{1}{3}.\dfrac{3}{6}.\dfrac{5}{9}.\dfrac{7}{12} + ...$

प्रदर्शित कीजिए $\sqrt{3} = 1 + \dfrac{1}{3} + \dfrac{1}{3}.\dfrac{3}{6} + \dfrac{1}{3}.\dfrac{3}{6}.\dfrac{5}{9} + \dfrac{1}{3}.\dfrac{3}{6}.\dfrac{5}{9}.\dfrac{7}{12} + ...$

Sol. $\sqrt{3} = \left(\dfrac{1}{3}\right)^{-1/2} = \left(1-\dfrac{2}{3}\right)^{-1/2}$

$= 1 + \dfrac{1}{2} . \dfrac{2}{3} + \dfrac{1}{2}.\left(\dfrac{3}{2}\right).\left(\dfrac{2}{3}\right)^2 . \dfrac{1}{2} + \dfrac{1}{2}.\dfrac{3}{2}.\dfrac{5}{2}.\left(\dfrac{2}{3}\right)^3 . \dfrac{1}{3!} + ...$

$= 1 + \dfrac{1}{3} + \dfrac{1}{3}\left(\dfrac{3}{6}\right) + \dfrac{1}{3}\left(\dfrac{3}{6}\right)\left(\dfrac{5}{9}\right) + ...$

30. If $(1 + x)^n = C_0 + C_1x + C_2x^2 + ... + C_nx^n$, show that for $m \ge 2$

$C_0 - C_1 + C_2 - .... + (-1)^{m-1}C_{m-1} = (-1)^{m-1}\ {}^{n-1}C_{m-1}$.

यदि $(1 + x)^n = C_0 + C_1x + C_2x^2 + ... + C_nx^n$ हो, तो प्रदर्शित कीजिए कि $m \ge 2$ के लिए

$C_0 - C_1 + C_2 - .... + (-1)^{m-1}C_{m-1} = (-1)^{m-1}\ {}^{n-1}C_{m-1}$.

Sol. $(x-1)^n = C_0x^n - C_1x^{n-1} + C_2x^{n-2} - C_3x^{n-3} + ...+ (-1)^{m-1}C_{m-1}x^{n-m+1} + ...$

$\Rightarrow \quad \dfrac{1-x^m}{1-x} = 1 + x + x^2 + ...+ x^{m-1}$

$C_0 - C_1 + C_2 - C_3 + ...+(-1)^{m-1}C_{m-1}$

$=$ Co-efficient of x^n in $(x-1)^n \left(\dfrac{1-x^m}{1-x}\right)$

$=$ Co-efficient of x^n in $(x^m-1)(x-1)^{n-1}$

$=$ Co-efficient of x^{n-m} in $(x-1)^{n-1}$

$= (-1)^{m-1}\ {}^{n-1}C_{m-1}$

Hindi $(x-1)^n = C_0x^n - C_1x^{n-1} + C_2x^{n-2} - C_3x^{n-3} + ...+ (-1)^{m-1}C_{m-1}x^{n-m+1} + ...$

$\Rightarrow \quad \dfrac{1-x^m}{1-x} = 1 + x + x^2 + ...+ x^{m-1}$

$C_0 - C_1 + C_2 - C_3 + ...+(-1)^{m-1}C_{m-1}$

$= (x-1)^n\left(\dfrac{1-x^m}{1-x}\right)$ में x^n का गुणांक

$= (x^m-1)(x-1)^{n-1}$ में x^n का गुणांक $= (x-1)^{n-1}$ में x^{n-m} का गुणांक

$= (-1)^{m-1}\ {}^{n-1}C_{m-1}$

31. If $(1+x)^n = C_0 + C_1x + C_2x^2 + ... + C_n x^n$, then show that the sum of the products of the C_i's taken two at a time, represented by $\underset{0 \le i < j \le n}{\sum\sum} C_i C_j$ is equal to $2^{2n-1} - \frac{2n!}{2(n!)^2}$.

यदि $(1+x)^n = C_0 + C_1x + C_2x^2 + ... + C_n x^n$, तब दर्शाओं कि दो C_i's को एक साथ लेने पर उनके गुणनफलनों का योग जोकि $\underset{0 \le i < j \le n}{\sum\sum} C_i C_j$ द्वारा प्रदर्शित होता है $2^{2n-1} - \frac{2n!}{2(n!)^2}$ के बराबर है।

Sol. $(1+x)^n = C_0 + C_1x + ... + C_n x^n$

$S = \sum\sum C_iC_j = \frac{1}{2}\left[(C_0 + C_1 + C_2 + ...+ C_n)^2 - (C_0^2 + C_1^2 + C_2^2 + ...+ C_n^2)\right]$

$\Rightarrow 2S = 2^{2n} - {}^{2n}C_n \qquad \Rightarrow S = 2^{2n-1} - \frac{{}^{2n}C_n}{2}$

32. If $a_0, a_1, a_2,.....$ be the coefficients in the expansion of $(1 + x + x^2)^n$ in ascending powers of x, then prove that :

(i) $a_0 a_1 - a_1 a_2 + a_2 a_3 -.... = 0$

(ii) $a_0a_2 - a_1a_3 + a_2a_4 -..... + a_{2n-2}\, a_{2n} = a_{n+1}$

(iii) $E_1 = E_2 = E_3 = 3^{n-1}$; where $E_1 = a_0 + a_3 + a_6 +...$; $E_2 = a_1 + a_4 + a_7 +...$ & $E_3 = a_2 + a_5 + a_8 +...$

यदि $a_0, a_1, a_2,.....$, $(1 + x + x^2)^n$ के प्रसार में x की बढ़ती हुई घातों के गुणांक हैं, तो सिद्ध करो कि :

(i) $a_0 a_1 - a_1 a_2 + a_2 a_3 -.... = 0$

(ii) $a_0a_2 - a_1a_3 + a_2a_4 -..... + a_{2n-2}\, a_{2n} = a_{n+1}$

(iii) $E_1 = E_2 = E_3 = 3^{n-1}$; जहाँ $E_1 = a_0 + a_3 + a_6 +...$; $E_2 = a_1 + a_4 + a_7 +...$ एवं $E_3 = a_2 + a_5 + a_8 +...$

Sol. $(1 + x + x^2)^n = a_0 + a_1x + a_2x^2 + a_3x^3 + ...+ a_{2n}x^{2n}$

taking $-\frac{1}{x}$ in place of x.

$\left(1 - \frac{1}{x} + \frac{1}{x^2}\right)^n = a_0 - \frac{a_1}{x} + \frac{a_2}{x^2} - \frac{a_3}{x^3} + ...$

(i) $\therefore$ $a_0a_1 - a_1a_2 + a_2a_3$ ....= coefficient of x in $(1 + x + x^2)^n \left(\frac{x^2 - x + 1}{x^2}\right)^n$

= coefficient of x in $\frac{1}{x^{2n}}(x^4 + x^2 + 1)^n$

= 0

(ii) $a_0 a_2 - a_1a_3 + a_2a_4$ .... = coeff. of x^2 in $\frac{1}{x^{2n}}(x^4 + x^2 + 1)^n$

$= a_{n+1}$

(iii) putting $x = 1, \omega$ & ω^2 respectively we get

$3^n = a_0 + a_1 + a_2 + a_3 + a_4 + a_5 + a_6 + ...$ (1)

$0 = a_0 + a_1\omega + a_2\omega^2 + a_3\omega^3 + a_4\omega^4 + a_5\omega^5 + a_6\omega^6 + ...$ (2)

$0 = a_0 + a_1\omega^2 + a_2\omega^4 + a_3\omega^6 + a_4\omega^8 + a_5\,\omega^{10} + a_6\omega^{12} + ...$ (3)

on adding

$3^n = 3(a_0 + a_3 + a_6 + ...)$

$\Rightarrow \quad E_1 = 3^{n-1}$

(1) + ω^2 (2) + ω(3) gives

$3^n = 3(a_1 + a_4 + a_7 + ...)$

$\Rightarrow \quad E_2 = 3^{n-1}$

Similarly

$(1) + \omega(2) + \omega^2(3)$ gives

$E_3 = 3^{n-1}$.

Hindi. $(1 + x + x^2)^n = a_0 + a_1x + a_2x^2 + a_3x^3 + ...+ a_{2n}x^{2n}$

x के स्थान पर $-\frac{1}{x}$ रखने पर

$$\left(1-\frac{1}{x}+\frac{1}{x^2}\right)^n = a_0 - \frac{a_1}{x}+\frac{a_2}{x^2}-\frac{a_3}{x^3}+......$$

(i) $\therefore \quad a_0a_1 - a_1a_2 + a_2a_3 ....= (1 + x + x^2)^n \left(\frac{x^2-x+1}{x^2}\right)^n$ में x का गुणांक

$= \frac{1}{x^{2n}}(x^4 + x^2 + 1)^n$ में x का गुणांक

$= 0$

(ii) $a_0\, a_2 - a_1a_3 + a_2a_4 .... = \frac{1}{x^{2n}}(x^4 + x^2 + 1)^n$ में x^2 का गुणांक

$= a_{n+1}$

(iii) $x = 1, \omega$, और ω^2 क्रमशः रखने पर

$3^n = a_0 + a_1 + a_2 + a_3 + a_4 + a_5 + a_6 + ... (1)$

$0 = a_0 + a_1\omega + a_2\omega^2 + a_3\omega^3 + a_4\omega^4 + a_5\omega^5 + a_6\omega^6 + ... (2)$

$0 = a_0 + a_1\omega^2 + a_2\omega^4 + a_3\omega^6 + a_4\omega^8 + a_5\,\omega^{10} + a_6\omega^{12} + ... (3)$

(1), (2) व (3) को जोड़ने पर

$3^n = 3(a_0 + a_3 + a_6 + ...)$

$\Rightarrow \quad E_1 = 3^{n-1}$

$(1) + \omega^2 (2) + \omega(3)$ से प्राप्त होता है–

$3^n = 3(a_1 + a_4 + a_7 + ...)$

$\Rightarrow \quad E_2 = 3^{n-1}$

इसी प्रकार

$(1) + \omega(2) + \omega^2(3)$ से प्राप्त होता है–

$E_3 = 3^{n-1}$.